Exercise 25: Applying the epsilon net theorem (4 Points)

Consider the set system $(X, \mathcal{F})$ where $X = [0, 1]^2$ is the unit square and
\[ \mathcal{F} = \{ x \cap B_{0,1}(x) \mid x \in X \}. \]
Here, $B_{0,1}(x) := \{ y \mid d(x,y) \leq 0.1 \}$ is a circle of radius 0.1, centered in $x$. $d(\cdot,\cdot)$ is the Euclidean distance. The measure $\mu(A)$ of set $A \subset X$ equals the area covered by $A$.

For any value $0 < \varepsilon \leq 0.01\pi$,

a) Use the epsilon net Theorem to obtain an upper bound on the size of an $\varepsilon$ net for $(X, \mathcal{F})$. Check the requirements for applying the epsilon net Theorem, i.e. determine the value $\text{dim}_{\text{VC}}(\mathcal{F})$ and the value of the constant $C$ as in the proof of the epsilon net Theorem in the lecture.

b) Construct an $\varepsilon$ net for $(X, \mathcal{F})$ and compare its size with the value obtained in a).
Exercise 26: Random variables (4 Points)

The variance \( \text{Var}(X) \) of a random variable \( X \) is defined as

\[
\text{Var}(X) := \mathbb{E}((X - \mathbb{E}(X))^2)
\]

where \( \mathbb{E}(\cdot) \) denotes the expected value. Two random variables \( X, Y \), are called \textit{independent}, if for all (measurable) sets, \( A, B \), the equality

\[
P(X \in A \land Y \in B) = P(X \in A) \cdot P(Y \in B)
\]

is fulfilled. They are called \textit{uncorrelated}, if

\[
\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)
\]

holds.

a) Give a simple example of two random variables which are independent, but not uncorrelated.

b) Show that if \( X \) and \( Y \) are independent random variables, which attain finitely many values only, then \( X \) and \( Y \) are also uncorrelated.

c) Prove that if \( X \) and \( Y \) are two uncorrelated random variables, then
\[
\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)
\]
holds.
Exercise 27: Packings and transversals (4 Points)

Let natural numbers $k \leq n$ be given. We consider the basic set $X = \{1, \ldots, n\}$ and the set system

$$\mathcal{F} := \{Y \subseteq X \mid |Y| = k\}.$$ 

A subset $T \subseteq X$ is called a transversal of $\mathcal{F}$ if it intersects all the (non-empty) sets of $\mathcal{F}$. The transversal number, denoted by $\tau(\mathcal{F})$, is the smallest possible cardinality of a transversal of $\mathcal{F}$. The packing number of $\mathcal{F}$, denoted by $\nu(\mathcal{F})$, is the maximum cardinality of a system of pairwise disjoint sets in $\mathcal{F}$.

$$\nu(\mathcal{F}) = \sup \{|M| : M \subseteq \mathcal{F}, M_1 \cap M_2 = \emptyset \text{ for all } M_1, M_2 \in M, M_1 \neq M_2\}$$

For a finite set $X$, as in this exercise, we define a fractional transversal for $\mathcal{F}$ to be a function $\phi : X \mapsto [0, 1]$ such that for each $S \in \mathcal{F}$, we have $\sum_{x \in S} \phi(x) \geq 1$. The size of a fractional transversal $\phi$ is $\sum_{x \in X} \phi(x)$, and the fractional transversal number $\tau^*(\mathcal{F})$ is the infimum of the sizes of fractional transversals. A fractional packing for $\mathcal{F}$ is a function $\psi : \mathcal{F} \mapsto [0, 1]$ where for each $x \in X$, we have $\sum_{S \in \mathcal{F}, x \in S} \psi(S) \leq 1$. The size of a fractional packing $\psi$ is $\sum_{S \in \mathcal{F}} \psi(S)$, and the fractional packing number $\nu^*(\mathcal{F})$ is the supremum of the sizes of all fractional packings for $\mathcal{F}$.

For the given base set $X$ and set system $\mathcal{F}$, determine the transversal number $\tau(\mathcal{F})$, the packing number $\nu(\mathcal{F})$, and their fractional variants $\tau^*(\mathcal{F})$ and $\nu^*(\mathcal{F})$. 