Exercise 7: Proof details Two-Squares-Theorem (4 Points)

1. For \( p = 17 \), present the corresponding values of \( q, a \) and \( b, i \) and \( j \) in the proof of the Two-Squares-Theorem (Theorem 11). Finally \( p = a^2 + b^2 \) for \( a, b \in \mathbb{Z} \) has to be fulfilled.

2. Prove the following statement: For the factor ring \( \mathbb{Z}_p \) for a prime \( p \) only \( a = \bar{a} \) and \( a = -\bar{a} \) gives a solution for \( a^2 = \bar{a} \).
   (You can make use of the following statement: \( p|ab \Rightarrow p|a \) or \( p|b \).)
Exercise 8: Minkowskis Theorem (4 Points)

- Present an argument that the Minkowski Theorem (Theorem 7) actually says that 2 lattice points different from the origin will be inside the set $C$.

- Argue that the boundedness of the set $C$ is not a necessary condition of Theorem 7. Give an example for an unbounded set $C$ that fulfills the conditions of Theorem 7 for $\mathbb{R}^2$.

Exercise 9: Application of Minkowskis Theorem (4 Points)
Consider the regular $(5 \times 5)$ lattice around the origin. Calculate the required expansion (radius $r$) of the trees at the lattice points so that any line $Y = aX$ hits at least one of the trees. Do the calculation in the following ways:

1. Calculate the radius $r$ directly and precisely by considering the corresponding circles and lines.
   (W.l.o.g. only two cases have to be considered!)

2. Make use of the Minkowski Theorem and compute a non-trivial radius $r$ that fulfills the requirement.

Figure 1: The regular $(5 \times 5)$ grid. The line passes the circles.