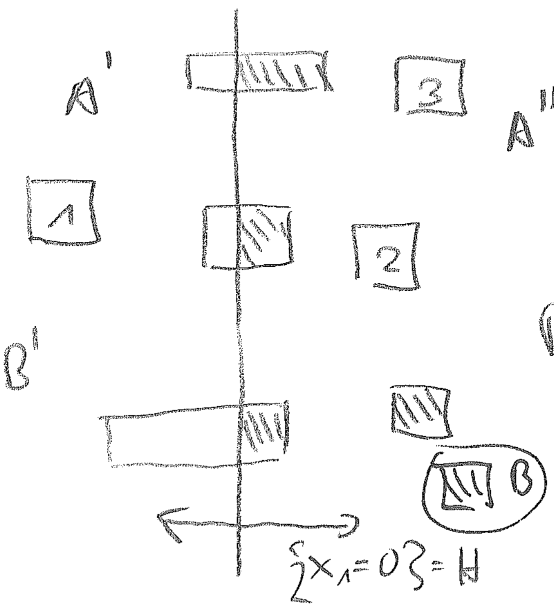
$k > 2$ w.l.o.g $|A| \geq 2$

Let $H$ be hyperplane in $\mathbb{R}^d$ that subdivides at least two bricks of $A$

w.l.o.g. $H = \{x_1 = 0\}$




Translate $B$ to $B', B''$

So that:

$\mathrm{vol}(A') = s \, \mathrm{vol}(A)$
$\mathrm{vol}(A'') = (1-s) \, \mathrm{vol}(A)$

By translation: $\mathrm{vol}(B') = s \, \mathrm{vol}(B)$
$\mathrm{vol}(B'') = (1-s) \, \mathrm{vol}(B)$

$A', A''$ $B', B''$ have the same volume relationship

[Translation changes position of $B$ but not $\mathrm{vol}(A+B)$, $\mathrm{vol}(B)$]

$$|\mathrm{bricks}(A')| + |\mathrm{bricks}(B')| < k \overset{\text{Ind. hyp.}}{\Rightarrow} \mathrm{vol}(A')^{\frac{1}{d}} + \mathrm{vol}(B')^{\frac{1}{d}} \leq \mathrm{vol}(A'+B')^{\frac{1}{d}}$$

$$|\mathrm{bricks}(A'')| + |\mathrm{bricks}(B'')| < k \Rightarrow \mathrm{vol}(A'')^{\frac{1}{d}} + \mathrm{vol}(B'')^{\frac{1}{d}} \leq \mathrm{vol}(A''+B'')^{\frac{1}{d}}$$

$$\mathrm{vol}(A+B) = \mathrm{vol}\left((A' \cup A'') + (B' \cup B'')\right)$$

$$\geq \mathrm{vol}(A'+B') + \mathrm{vol}(A''+B'')$$

$A'+B'$ in $\{x_1 \leq 0\}$
$A''+B''$ in $\{x_1 \geq 0\}$

(contains $A'+B'$ and $A''+B''$ but also other sums, $A'+B'$, $A''+B''$ do not overlap)

$(\,)^d$ and Ind. hypoth.

$$\geq \left(vol(A')^{\frac{1}{d}} + vol(B')^{\frac{1}{d}}\right)^d + \left(vol(A'')^{\frac{1}{d}} + vol(B'')^{\frac{1}{d}}\right)^d$$

$$= \left(s^{\frac{1}{d}}\, vol(A)^{\frac{1}{d}} + s^{\frac{1}{d}}\, vol(B)^{\frac{1}{d}}\right)^d + \left((1-s)^{\frac{1}{d}} vol(A)^{\frac{1}{d}} + (1-s)^{\frac{1}{d}} vol(B)^{\frac{1}{d}}\right)^d$$

$$= s\left(vol(A)^{\frac{1}{d}} + vol(B)^{\frac{1}{d}}\right)^d + (1-s)\left(vol(A)^{\frac{1}{d}} + vol(B)^{\frac{1}{d}}\right)^d$$

$$= \left(vol(A)^{\frac{1}{d}} + vol(B)^{\frac{1}{d}}\right)^d.$$

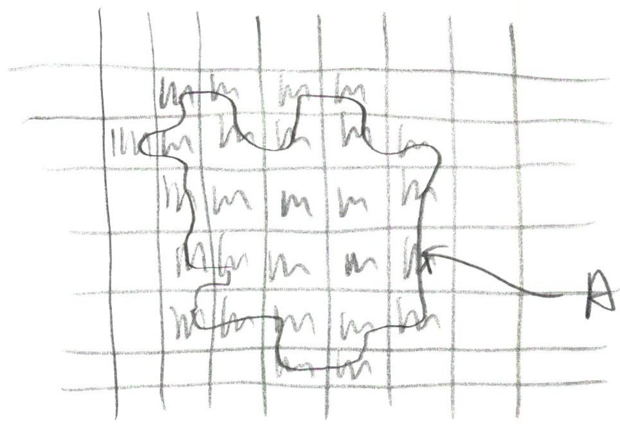
□

Statement holds for all brick sets ✓

Let $A, B$ compact sets Lebesgue-measurable set

( Measure theory ⇝ Standard way of assigning volume to sets in $\mathbb{R}^d$
Volume of boxes )

Consider set $A$ (analogously for $B$)
Subdivision of $\mathbb{R}^d$ into "boxes" of side-length $\frac{1}{2^k}$ (hypercubes)



$A_k$
Set of hypercubes of side length $\frac{1}{2^k}$ intersecting with $A$

Approximation ✓

Measure theory: $\bigcap_{i=1}^{\infty} A_i =: \bigcap_k A_k = A$

$\Rightarrow \lim_{k \to \infty} vol(A_k) = vol(A)$ (also for $B, B_k$)

(Final observation)



## Lemma 35

$$\bigcap_{\ell} (A_\ell \oplus B_\ell) = A \oplus B$$

**Proof:** "$\supseteq$" trivial any $A_\ell \oplus B_\ell$ is supersets of $A \oplus B$

"$\subseteq$" $x \in A_\ell \oplus B_\ell$ for all $\ell$

$\Rightarrow$ there are series $\underset{\in A_k}{\{a_\ell\}}$ and $\underset{\in B_k}{\{b_\ell\}}$

with $x = a_\ell + b_\ell \quad \forall \ell$

$\Rightarrow$ $\{a_\ell\}$ $\{b_\ell\}$ have sub series $\overline{\{a_\ell\}}$ $\overline{\{b_\ell\}}$

Bolzano-Weierstrass

that converge (bounded set)

$\overline{\{a_\ell\}} \to a \qquad \overline{\{b_\ell\}} \to b$

$A, B$ closed (compact) $\quad a \in A, \; b \in B$

$\Rightarrow x = a + b$ $\square$

**Now Theorem 33:** For general $A, B$

$$\mathrm{vol}(A+B)^{\frac{1}{d}} = \lim_{\ell \to \infty} \mathrm{vol}(A_\ell + B_\ell)^{\frac{1}{d}} \geq \lim_{\ell \to \infty} \left( \mathrm{vol}(A_\ell)^{\frac{1}{d}} + \mathrm{vol}(B_\ell)^{\frac{1}{d}} \right)$$

Lemma 35, Measure Theory

$A \oplus B = \bigcap_{\ell=0}^{\infty} (A_\ell \oplus B_\ell)$

Bricks-Lemma $(\ast)$

$$= \mathrm{vol}(A)^{\frac{1}{d}} + \mathrm{vol}(B)^{\frac{1}{d}}$$

indep./ Measure-Theory

$\square$

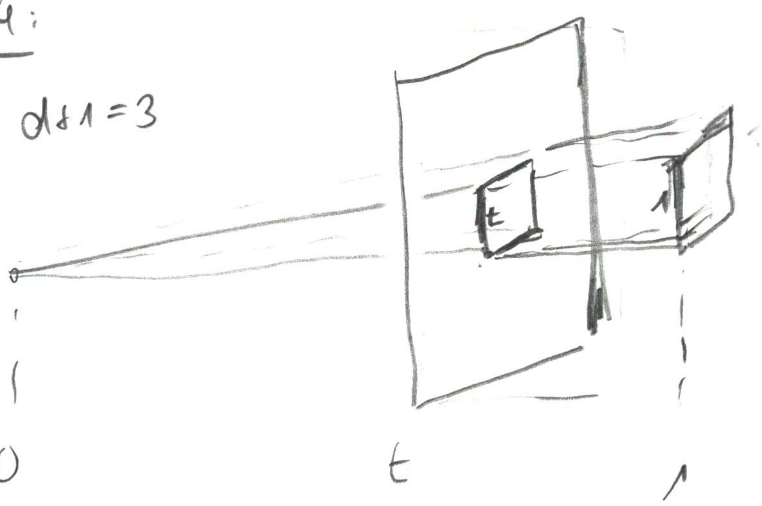
Observation: $d \geq 3$

$V(t)^{\frac{1}{d}}$ always concave

$V(t)$ not in general.

Example:

$d+1=3$




$d$-dim. cube

$$V(t) = C \cdot t^d \quad \underline{\text{convex}} \qquad V(t)^{\frac{1}{d}} \quad \underline{\text{concave}}$$

$$= C^{\frac{1}{d}} \cdot t \quad \text{(real) linear!}$$

Korollar 36: Let $A, B \subseteq \mathbb{R}^d$ compact sets.

$$\sqrt{\operatorname{vol}(A) \cdot \operatorname{vol}(B)} \leq \operatorname{vol}\left(\frac{A \oplus B}{2}\right)$$

Proof:

$$\operatorname{vol}\left(\frac{A \oplus B}{2}\right)^{\frac{1}{d}} \underset{\text{Brunn/Mink.}}{\geq} \operatorname{vol}\left(\frac{A}{2}\right)^{\frac{1}{d}} + \operatorname{vol}\left(\frac{B}{2}\right)^{\frac{1}{d}}$$

$$= \frac{1}{2}\left(\operatorname{vol}(A)^{\frac{1}{d}} + \operatorname{vol}(B)^{\frac{1}{d}}\right)$$

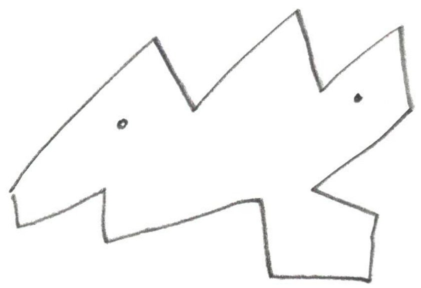
$$\underset{\text{arithm.} \geq \text{geom.}}{\geq} \left(\operatorname{vol}(A) \cdot \operatorname{vol}(B)\right)^{\frac{1}{2d}} \qquad \text{Now } (\,)^d$$

$\square$

(Vapnik – Chervonenkis)

# Art Gallery + VC-dimension

**Problem:** Art Gallery (simple area)



polygon

n vertices

**Q:** Minimum number of stationary guards that "sees" full polygon?

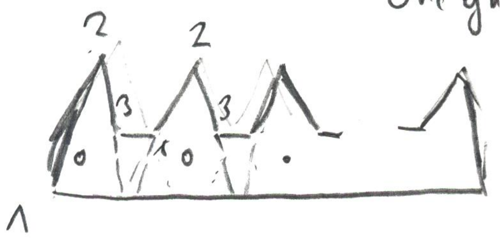
NP-hard Aggarwal, Lee Lin 83/86

**Theorem 36** (Chvátal) $\lfloor \frac{n}{3} \rfloor$ guards are sufficient

Sometimes $\lfloor \frac{n}{3} \rfloor$ guards are necessary

**Proof:** Second part:

One guard for each peak!



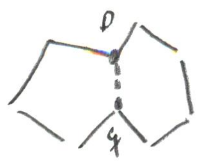

3m vertices, m guards

First part: Triangulation of polygon

n-2 triangles

n-3 diagonals

Induction, existence of diagonals




Line segment pq that cuts polygon

Triangulation: Maximum number of diagonals

**Lemma 37** In any triangulation of a single polygon P there is at least one triangle whose boundary has at most one diagonal

$n-2$ triangles, outer boundary edges belong to unique triangles.

For $n>3$ any triangle has at most 2 boundary edges. Disjoint boundary edges.

($n$ edges)

If there is one with 2 boundary edges $\Rightarrow$ claim ✓

If there is none with 2 boundary edges $\Rightarrow$ at most $n-2$ edges ↯ $n$ edges of P (2 such triangles)

Colorability: with three colors, place colors at the vertices

Graph structure: The vertices of an edge do not have the same colors, min. number of colors

Triangulation of P: 3 colors are enough

Coloring $\rightsquigarrow$ Take the color with the smallest number $n$ vertices

⌈ Does not hold for general triangulations, degree! ⌋ 

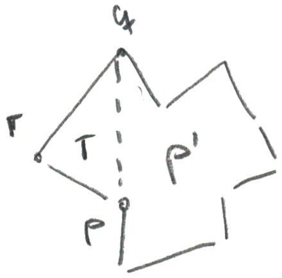
$\#$ Guards $\leq \lfloor \frac{n}{3} \rfloor$ (integer)

Remains to show: 3 colors are enough for a triangulation

Induction: $n=3$ ✓

$n>3$ Choose triangle with one diagonal

Split:




Use ind. hyp. for P'

3 colors p and q have different colors

r gets the third color.

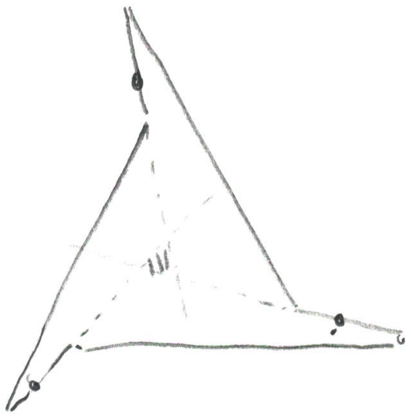
Ind. step ✓

□

## Some simple observations

P simple polygon stationary guards sees $\partial P$

$\not\Rightarrow$ guards do not see the whole polygon

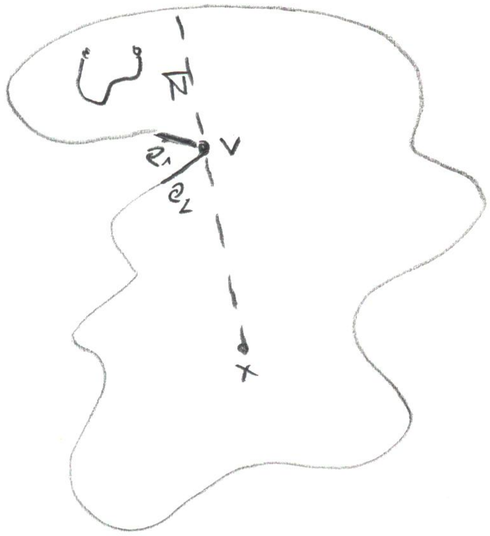


But: A mobile guard along path $\pi$ sees the boundary

$\Rightarrow$ the guard sees the whole polygon

Proof:

Let $\pi$ be the guarding path




Assume that

$\pi$ does not "see" $x$

$\Rightarrow \pi \cap vis_P(x) = \emptyset$

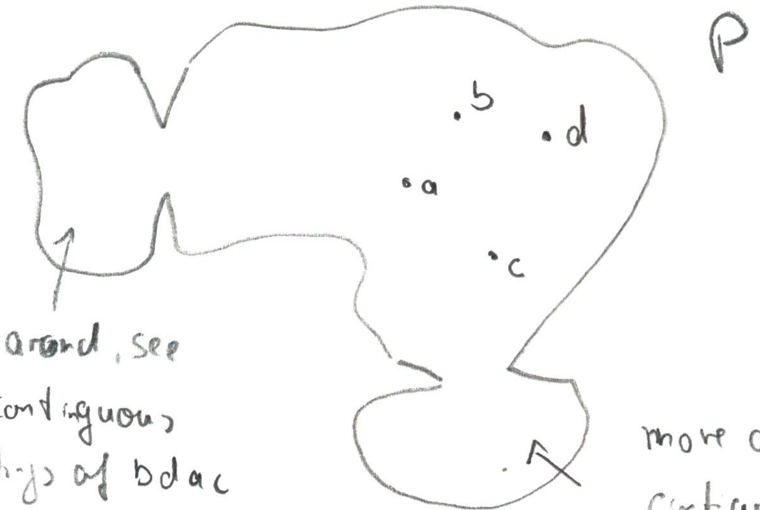
$\Rightarrow \pi$ "behind" vertex $v$

does not see $e_2$

$\square$

Walking around in a polygon with $n$ points:

From each position some points can be seen, some not.




move around, see all contiguous substrings of bdac

move around, see all contiguous substrings of abcd

There are positions in P so that any subset of $\{a, b, c, d\}$ can be precisely seen.

Only b, c, a, d only bd, ac, ab, cd and so on ...

Q: What is the maximum number d of points for which such examples can be constructed

| | | |
|---|---|---|
| Matoušek | $5 \le d < \infty$ | |
| Valtr | $6 \le d \le 23$ | |
| Gilbers, Klein 2011 | $d \le 14$ | |

Position in P from which point a can be seen is $\mathrm{Vis}_P(a)$

Analogous question!

Theorem 38 (Gilbers, Klein)

$\forall P \; \forall$ 15 points $a_1, a_2, \ldots, a_{15} \in P$ not all subsets of $\{ \mathrm{vis}(a_i) \mid 1 \le i \le 15 \}$ can be stabbed

Stabbing: Example line segments / interval,


stabs $l_1, l_2, l_3$ stabs $l_4$ $l_5$ $l_6$

Intervals: no more than 2


Sketch of the proof: Idea replace "ugly" vispolygons by simple wedges.

Nice results for wedges.

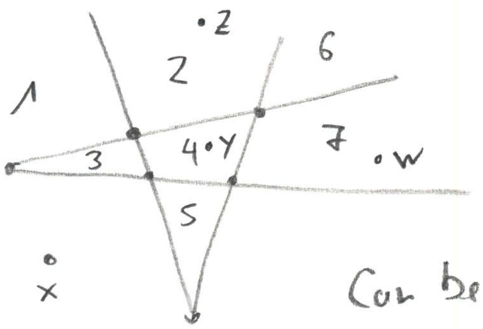
**Lemma 39** (Isler et al.)

In an arrangement of $\geq 6$ wedges not every subset can be stabbed.

**Proof:** Euler Formula $(v - e + f = c + 1)$

Arrangement of $n$ wedges has $n + x + 1$ cells (Exercise)!
where $x$ = # of half line intersections

Ex.




$x = 4$ $\quad n + x + 1 = 7$
$n = 2$

Can be stabbed
$x, y, z, w$

$n = 6 \Rightarrow \max 4 \times \binom{n}{2} = 4 \cdot \binom{6}{2} = 4 \cdot \frac{6!}{2!\,4!} = 4\,\frac{6 \cdot 5}{2} = 60$ intersections

$\Rightarrow \max 6 + 60 + 1 = 67$ cells

Enough cells for the stabbing of $2^6 = 64$ possibilities

But some have the same subsets $\Rightarrow$ not enough cells for all subsets

Some subsets counting

(i) every wedge properly crossed by all other wedges

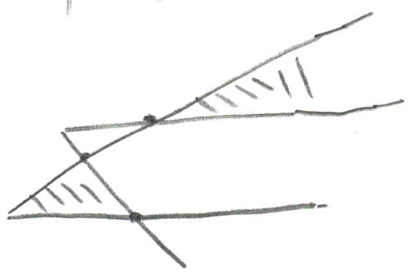
W  First and last cell contained only in W (remove one cell)

(ii) Remaining $n'$ wedges not properly crossed by the other wedges (but crossed by all)

Number of cells saved?

$\# |W \cap W'| = 1$ 3 intersections missing
$\# |W \cap W'| = 2$ 2 intersections missing
} For each $W, W'$ at least one cell less

$\# |W \cap W'| = 3$



1 intersection missing + 1 cell redundant } For each $W, W'$ at least one cell less

$\Rightarrow$ $n'$ cells redundant or less $\checkmark$!

alltogether $67 - 6 < 64$ possibilities

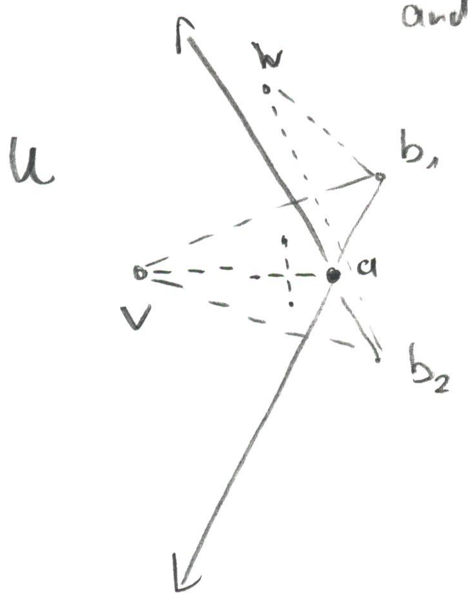
$\square$

Proof of Theorem 38 Substitute vispolygons by wedges $\checkmark$!

Suppose $\exists P, S \subset P$ $|S| = 15$ and $S$ is "shattered"

$\Rightarrow \forall T \subseteq S \ \exists v_T \in P: \ vis(v_T) \cap S = T$

Let $a, b_1, b_2 \in S$ $v = v_T$ where $a \notin T$, $b_1, b_2 \in T$ and $a \in \text{tria}(v, b_1, b_2)$



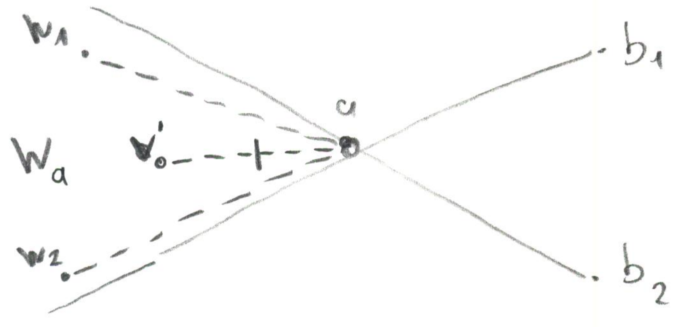

<u>Claim V:</u> w sees $b_1, b_2$

$\Rightarrow w \in U$ (wedge)

Proof: Otherwise $\overline{va}$ "surrounded", not a simple polygon

Let $w_1, w_2$ be the outermost points of $V := \{v_T \mid T \in S\}$ that sees $a, b_1, b_2$ ($\exists$ by assump. $\in U$ by Claim V)

Let $W$ denote the wedge formed by $a, w_1, w_2$.




<u>Def:</u> $V_{\{b_1, b_2\}} := \{v_T \in V \mid \{b_1, b_2\} \subseteq T\}$

<u>Claim VI:</u> $V_{\{b_1,b_2\}} \cap W_a = V_{\{b_1,b_2\}} \cap \text{vis}(a)$

"$\supseteq$" Claim V, Def. of $W_a$

"$\subseteq$" Let $v' \in V_{\{b_1,b_2\}} \cap W_a$

Assume $\Rightarrow$ $v'$ does not see $a$

$\Rightarrow$ $\overline{v'a}$ is surrounded ↯

$\Rightarrow$ $v'$ sees $a$ $v' \in vis(a)$ □

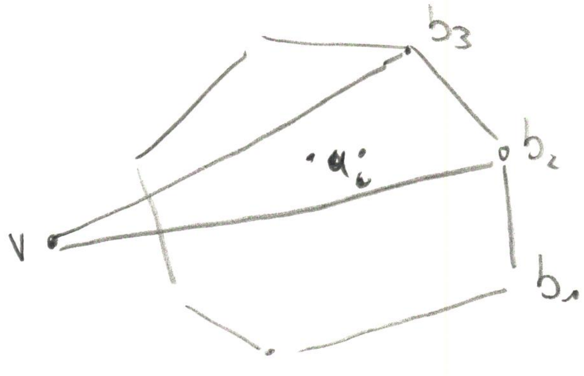
Proof of Theorem consists of 2 parts

$\boxed{\text{I}}$ $\leq 5$ points inside $ch(s)$

$\boxed{\text{II}}$ $\leq 9$ points on $ch(s)$

Proof only for $\boxed{\text{I}}$

Assume $a_1, \dots, a_6$ inside $ch(s)$, $b_1, \dots, b_9$ on $ch(s)$




Let $v := V_{\{b_1, \dots, b_9\}}$

Situation of Claim IV $\Rightarrow$ $\exists$ wedge $W_i$: for $i = 1, \dots, 6$

$V_{\{b_2,b_3\}} \cap W_i = V_{\{b_2,b_3\}} \cap vis(a_i)$

$\Rightarrow$

$$\forall i: V_{\{b_1,\dots,b_q\}} \cap W_i = V_{\{b_1,\dots,b_q\}} \cap vis(a_i)$$

By assumption: $\forall A \subseteq \{a_1,\dots,a_6\} \ \exists \ v_{B\cup A} \in V$

$B := \{b_1,\dots,b_q\}$

$\Rightarrow$ $v_{B\cup A}$ stabs exactly the vispolygons of $A$

$\Rightarrow$ $v_{B\cup A}$ stabs exactly the wedges of $A$ $\lightning$

Lemma 39

Proof of $\boxed{\text{IV}}$ very complicated, similar arguments!

$\square$

# VC-Dimension

## Definition 40

Let $(X, \mathcal{F})$ be a set system of a ground set $X$.
$(\mathcal{F} \subseteq \mathcal{P}(X))$.

(i) $A \subseteq X$ is shattered by $\mathcal{F}$ $:\Leftrightarrow$

$$\forall B \subseteq A \ \exists F \in \mathcal{F} : B = F \cap A$$

(in other words $\mathcal{F}|_A := \{ F \cap A \mid F \in \mathcal{F} \} = \mathcal{P}(A)$ )

(ii) Vapnik-Chervonenkis dimension of $(X, \mathcal{F})$:

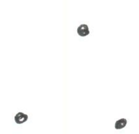
$\dim_{VC}(\mathcal{F}) :=$ max size of a shattered set
($= \infty$ possible)

**Examples:** 1) $X =$ all points in a simple polygon

$\mathcal{F} = \{ vis(a) \mid a \in X \} \Rightarrow \dim_{VC}(\mathcal{F}) \leq 14$
$\geq 6$.

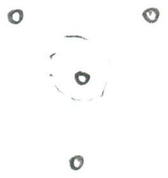
2) $X = \mathbb{R}^2$ $\mathcal{F} =$ Set of all closed half-spaces in $\mathbb{R}^2$

$\dim_{VC}(\mathcal{F}) = 3$



can be shattered by half spaces!

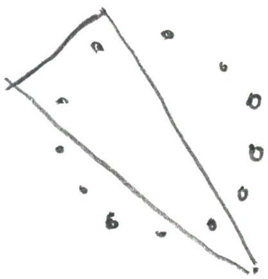
4 points



not separable

non-Convex convex positions

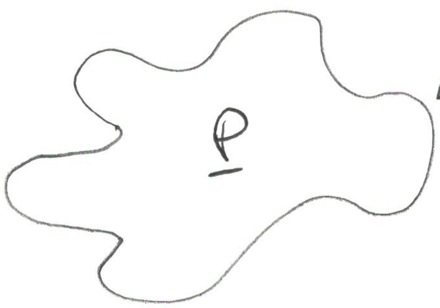
3) $X = \mathbb{R}^2$ $\quad \tilde{\mathcal{F}}$ = all convex sets in $\mathbb{R}^2$



The vertex set of any regular n-gon can be shattered

$n \to \infty$ $\qquad \dim_{VC}(\hat{\mathcal{F}}) = \infty$

Recap: Art gallery




Jordan curve (no holes)

Fixed number of Guards? $\quad Q_1$

No $\rightsquigarrow$ $\lfloor \frac{n}{3} \rfloor$ example

pikes

$Q_2 :=$ Assume that for any point $p$ in $P$ at least $\frac{1}{r}$ times vol($P$) can be seen.

Is there a bounded number (for example $r$) of guards that are sufficient?

**Proposition:** $P$ requires $w$ guards

$:\Longleftrightarrow$ set system $\left( \{ vis_P(p) \mid p \in P \} = \mathcal{F} \right)$

$(P, \mathcal{F})$ can be stabbed with $w$ points

**Definition 41** $\mathcal{F}$ set system over $X$. $(X, \mathcal{F})$

(i) A subset $T \subseteq X$ is called a transversal of $\mathcal{F}$ if $T$ intersect all sets of $\mathcal{F}$

$$\forall F \in \mathcal{F} \wedge F \neq \emptyset : F \cap T \neq \emptyset$$

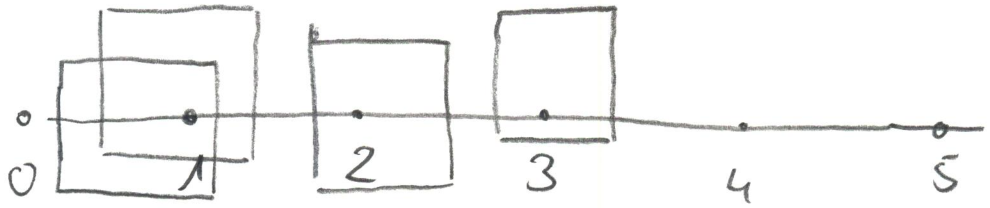
(ii) The transversal number $\tau(\mathcal{F})$ is the smallest possible cardinality of a transversal of $\mathcal{F}$.

- Such transversals always exists: Simply choose $T = X$

- Our example $\tau\left(\{ vis_P(p) \mid p \in P \}\right)$ smallest number of guards!

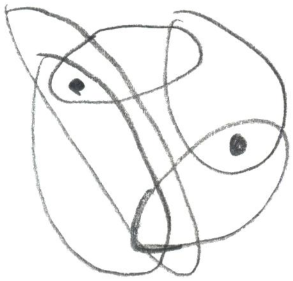
- Example $\tau(\mathcal{F}) = \infty$

$X = \mathbb{R}^2$ $\mathcal{F} = \{$ all closed unit squares that hit the $x$-axis $\}$




$\mathbb{Z}$ is a transversal

- Example:



$\tau = 2$

Many problem can be redefined by questions about transversals and its numbers

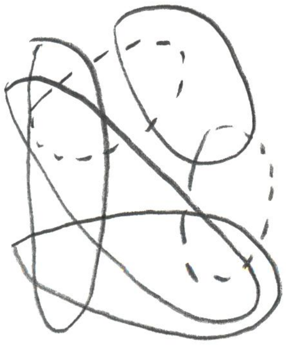
Another important paramete

Definition 42 $\mathcal{F}$ set system over $X$. $(X, \mathcal{F})$

(i) A subset $M \subseteq \mathcal{F}$ is called a packing (matching) $:\Leftrightarrow \forall F_1, F_2 \in M \Rightarrow F_1 \cap F_2 = \emptyset$

(ii) The packing number $\nu(\mathcal{F})$ is the maximal cardinality of a packing of $\mathcal{F}$.

- Ex.



$\nu = 2$