Proof:

We apply the randomized Min-Algorithm on

\[ w(p_1), w(p_2), \ldots, w(p_r) \]

Let \( T(p) \) denote the running time for computing \( w(p) \) (random variable)

\[ N(p) = \sum_{i=1}^{r} w(p_i) \text{ so to be computed (0-1 variable)} \]

We have:

\[ E\left( \sum_{i=1}^{r} N(p_i) \right) \leq \text{bound} \]

As before, choose uniformly at random (backwards) \( \stackrel{\rightarrow}{p_i} \)'s (some choice to be the minimum)

Now set:

\[ A(n) = \max_{|P| \leq n} E\left( T(p) \right) \text{ maximum expected running time w.r.t. Problem size } \leq n. \]

\( P \in T(n), |P| \leq n \)

\[ T(p) = \sum_{i=1}^{r} N(p_i) T(p_i) + O\left( r \cdot D(p_i) \right) \]

1 Task

\[ N(p_i), T(p_i) \text{ independent} \]

Computation of \( T(p_i) \) independent from the nonidentity of Computation.
\[ E(\bar{T}(p)) \]
\[ = \sum_{i=1}^{r} E(N(p_i)) E(\bar{T}(p_i)) + O(D((p_1)) \]
\[ \leq (\ln r + 1) \sum_{i=1}^{r} \frac{1}{\alpha |p_i|} + O(D((p_1))) \]

\[ \text{T is a constant.} \]

Holds if \( p_i \) with \( |p_i| \leq n \)

Therefore

\[ \hat{\bar{T}}(n) \leq (\ln r + 1) \hat{\bar{T}}(\alpha n) + R \cdot D(n) \text{ with constant } R \]

Show \( \hat{\bar{T}}(n) \leq C \cdot D(n) \text{ constant } C \)

2. Cases:

\[ (\ln r + 1) \alpha^\varepsilon < 1 \]

Case 2:

By induction \( \hat{\bar{T}}(n) \leq C \cdot D(n) \) holds

\[ \hat{\bar{T}}(n) \leq (\ln r + 1) \hat{\bar{T}}(\alpha n) + R \cdot D(n) \]

\[ \leq C \cdot D(\alpha n) \text{ by Ind. Hypothesis} \]

\[ \left[ \begin{array}{c} \frac{D(\alpha n)}{(\alpha n)^{\varepsilon}} \leq \frac{D(n)}{n^\varepsilon} \end{array} \right] \]

Monotonicity

\[ \leq (\ln r + 1) \alpha^\varepsilon \cdot C \cdot D(n) + R \cdot D(n) \]

\[ \leq C \cdot D(n) \text{ for } \frac{R}{\alpha^{1-(\ln r + 1) \alpha^\varepsilon}} \leq C \]
Case II: \(( \ln x + 1 ) e^x \geq 1 \)

Do recursion on \(P_1's\) existing \(V\)

\(r \rightarrow r^l\) \hspace{1cm} \(\alpha \rightarrow \alpha^l\)

\(r \rightarrow r^l\) \hspace{1cm} \(\alpha \rightarrow \alpha^l\)

\(\lim_{x \to 0} \ln (r^{l+1}) \alpha^l = 0 \quad (0 < 1 \text{ second form})\)

\(\ln (r^{l+1}) \alpha^l < 1 \text{ for some } l\)

\((r^l \text{ is the new } r)\)

Application to Diameter Problem

Decision: \(d(c) \leq t\) \hspace{1cm} \(d(n) = n \log n\)

\(\frac{n \log n}{n^2} \rightarrow 0\)

Generalization for decomposition into subproblems

\(U:\) Set of Vertices of \(C\)

\(A:\) Set of Edges of \(C\)

\(\text{Compute } d(U, A) := \sup \{ d(p, q) \mid p \in U, q \in A\}\)

no segment of \(A\) cuts \(pq\)
Decision problem: \( d(U, Q) \leq t \) in time \( O(n \log n) \)

Decomposition

\[ U = U_1 \cup U_2 \]
\[ Q = Q_1 \cup Q_2 \]

\[ d(U, Q) = \max \left\{ \sum d(U_i, Q_j) : i, j \right\} \]

4 subproblems of size \( \frac{1}{2} (|U| + |Q|) \)

\( t = 4 \)
\( \alpha = \frac{1}{2} \)

Apply Chazelle's technique.

Vertex/Vertex Problem Graph-Theoretical Dilation

Use the same approach.

\( \text{AVD} \) Trace the path and evaluate at the metrics.

Decision problem in \( O(n \log n) \).

Theorem 5: The (graph-theoretical and geometric) dilation of a polygonal chain can be computed in expected time \( O(n \log n) \).
Lower bounds on the computational time:

Geometric algorithm: \( O(n) \) \( O(n \log n) \) still unknown.

Graph theoretic algorithm: \( O(n \log n) \)

(Algorithmic decision tree model)

Lemma 6: The graph-theoretic algorithm of a polygonal chain has computational time \( O(n \log n) \).

Proof: Reduction of element uniqueness.

Element uniqueness: \( Y_1, \ldots, Y_n \in \mathbb{Z} \)

Does there exist \( i \neq j \) with \( Y_i = Y_j \)?

\( O(n \log n) \) [Yao 1989]

Compute \( \text{graph}(C) \) faster than \( O(n \log n) \)

\( \Rightarrow \) answer element uniqueness faster than \( O(n \log n) \)

Reduction of \( Y_1, \ldots, Y_n \rightarrow \) chain \( C \)

in \( O(n) \) time.
\[ P_{2i} = \left( \frac{2i}{2n} \right), \quad i = 1, 2, ..., n \]
\[ P_{2i-1} = \left( \frac{2i-1}{2n} \right), \quad i = 1, 2, ..., n-1 \]

\[ \bar{Y} = \max_{1 \leq i \leq n} Y_i \quad Y = \min_{1 \leq i \leq n} Y_i \]

**Idea:** Maximum dilation for points

If \( Y_i = Y_j \) \( \implies \) max dilation between points \( P_{2i-1} \) and \( P_{2j-1} \)
\[
\begin{align*}
|\mathbf{p}_{2i-1} \cdot \mathbf{p}_{2i-1}| &= \sqrt{\left(\frac{2\delta - 1}{2n} - \frac{2i-1}{2n}\right)^2 + (Y_i^g - Y_i^v)^2} \\
\sum &= \frac{2(\gamma-i)}{2n} \quad \text{if} \quad Y_i^g = Y_i^v \\
2 &> \sqrt{\left(\frac{2(\gamma-i)}{2n}\right)^2 + 1} \quad \text{if} \quad Y_i^g \neq Y_i^v
\end{align*}
\]

\[
\left| \prod_{i=1}^{p_{2i-1}} \right| > 2(\gamma - \bar{Y}) (8-i) + 2(8-i)
\]

\[
\left| \prod_{i=1}^{p_{2i-1}} \right| < 2(\gamma - \bar{Y} + \delta) \left(\frac{\delta-i}{2}\right)(8-i)
\]

\[
\bar{\gamma}_i \quad \bar{\gamma}_i + \delta \\
(\gamma - \bar{Y})
\]

\[
\begin{align*}
\prod_{i=1}^{(8-i)\text{ times}} 2(\gamma - \bar{Y} + \delta) + (8-i)\text{ times}
\end{align*}
\]
\[
\frac{2 \left( \gamma - \gamma + \frac{n^2}{2} + \frac{1}{2} \right) (\delta - \iota)}{\sqrt{\left( \frac{2(\delta - \iota)}{2\kappa} \right)^2 + 1}} \leq \frac{2(\gamma - \gamma + 1)(\delta - \iota)}{2\kappa} \]

\[
2 \left( \gamma - \gamma + \frac{n^2}{2} + \frac{1}{2} \right) \frac{n^2}{\left( \frac{1}{n^2} \right)^2 + 1} \leq 2(\gamma - \gamma + 1) \cdot n^2
\]

\[
\iff \frac{\left( \gamma - \gamma + \frac{n^2}{2} + \frac{1}{2} \right)^2}{(\gamma - \gamma + 1)^2} \leq \left( \frac{1}{n^2} \right)^2 + 1
\]

\[
\rightarrow 1
\]

\[
D(n) \downarrow x
\]

for \( \gamma \rightarrow 00 \)
Dilation between $P_2$ and $P_{2+j}$

Similarly

$\frac{2(4-\gamma + \frac{1}{2})(\delta - 1)}{\sqrt{(2(\delta - 1))^2 + \delta}}$

This means construct chain

$C(Y_1, Y_2, \ldots, Y_n)$ with complexity $\gamma$ in $O(n)$ time:

Apply log for graph-theoretic dilation:

Pair $(P_x, P_e)$ attains maximum

If $P_{x,Y} = P_{e,Y}$ then element uniqueness

$Y_1, \ldots, Y_n$ \text{ and } $Y_o$

\text{ is not}
Some remarks:

1) Worst bound for geometric dilation of a chain \( O(n) \) \( O(n \log n) \) ?? Open problem

2) Deterministic approximation (1+\( \epsilon \)) approx
\( O \left( \frac{1}{\epsilon} \cdot n \log n \right) \) Grimm et al.

3) Diameter of cycles (round edges), trees
\( O(n \log^{1.5} n) \) Difference:

Separation pair: Agarwal et al. 09

4) Diameter planes: Graph, note n^2 values
\( O(n^2 \left( \frac{\log \log n}{\log n} \right)^4) \) Wolff-Motwani 09