Probability of all three events

Formally Bernoulli-Experiment (Stochastics)

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

Events \( A, B \) on \( \Omega \) with \( A \cup B \subseteq \Omega \)

\[ P(\Omega) = 1 \]

\[ \Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

\[ \Rightarrow P(A \cap B) \geq P(A) + P(B) - 1 \]

Events (\( S_d \)) \( A, B, C \)

\[ A \text{ height} \leq C \log n \Rightarrow \frac{3}{4} \]

\[ B \text{ size} \leq C n \Rightarrow \frac{3}{4} \]

\[ C \text{ cust. time} \leq C n \log n \Rightarrow \frac{3}{4} \]

\[ P[A \cap B \cap C] \geq P[A] + P[B \cap C] - 1 \]

\[ \geq P[A] + P[B] + P[C] - 2 \]

\[ \Rightarrow 3 \cdot \frac{3}{4} - 2 = \frac{1}{4} \]
Now back to expected size/bright sunny time

Let segments that build a simple polygon:
(R Seidel CGTA 1991)

Localisation of new segments costly?

Idea: Call → use the next end-point, easy to localise.

Problem: Random choice is destroyed!

Solution: Random insert as before but

- View the claim from time to time
  in order to localise end-points.

This helps for pillows.

Lemma 33: For $1 \leq s \leq r \leq n$.

If $g$ is a query point for which the trapezoid
of $T(s, g)$ is known, we can localise $g$ in $O(s, g)$
in average (expected time)

$$O\left(\frac{1}{d_{10}} + \frac{1}{s + 2} + \cdots + \frac{1}{s}ight) \in O\left(\log \frac{n}{\delta}\right)$$
Proof: As in the proof of Theorem 49

backward analysis starts at \( \Gamma(s_3) \)

Probab. that \( S_{3,n} \) changes trapezoid of \( \Gamma(s_{3,n}) \) is \( \frac{4}{8 + 1} \)
and so on:

\[
\sum_{i=8}^{x} 3 \cdot \frac{4}{i} \leq O\left( \log \frac{8}{x} \right)
\]

Estimation of the cost of a trace \( \Gamma \). Trace SIR through \( \Gamma(R) \)

Lemma 54:

Let \( R \subseteq S \) be a subset of non-intersecting segments
(\( P \) is a simple polygon). Then the segments of SIR
start (on the average) at most \( O(|S| - |R|) \) intersections
with the vertical lines of \( \Gamma(R) \).

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Diagram showing traces and SIR lines.
Proof:

For $s \in \mathcal{W}$ let

$$
\deg(s, w) := \text{the total number of edges connected to vertices in $w$ that but $s$}
$$

$$
\sum_{s \in \mathcal{W}} \deg(s, w) \leq 4 |w|
$$

(any vertices gain at most 4 neighbors)

\[
\begin{align*}
\sum_{\begin{subarray}{c} r = 1 \\ |R| = k \end{subarray}} \sum_{\begin{subarray}{c} s \in S' \cap R' \end{subarray}} \deg(s, R') & \leq 4(r+1) \\
\sum_{\begin{subarray}{c} r = 1 \\ |R| = k \end{subarray}} \binom{n}{r} \binom{n}{r+1} & \leq 4(r+1)
\end{align*}
\]

\[
\frac{4(r+1) \binom{n}{r} \binom{n}{r+1}}{(r+1)! (n-r-1)! n!}
\]
Define $S_5 \left( \log^2 n, \log^* n \right)$

\[
\log^{(i)} n := \log \left( \log^{(i-1)} n \right),
\]
\(i\)-times,

\[
\log^* n := \text{Smallest } h \text{ so that } \log^{(h)} n \leq 1.
\]

(Remal from my very short journey)

Algorithm: Segments $S_1, \ldots, S_n$

Procedure: $1, \ldots, \frac{n}{\log n}, \ldots, \frac{n}{\log \log n}, \ldots, \frac{n}{\log(3)n}, \ldots, \frac{n}{\log(\log^* n)n}, \ldots, n$

Choose a random permutation randomly $S_1, \ldots, S_n$

For any part $\frac{n}{\log^{(s-1)} n}, \ldots, \frac{n}{\log s n}$

- insert segments with correct indices
- trace at the end of the phase DP through the last structure. Locate all endpoints of non-wasted segments.
Theorem 56

A trapezoidal decomposition and a query structure of a single polygon can be computed in expected $O(n \log^2 n)$ time. (instead of $O(n \log n)$ for arbitrary segments)

Proof:

$\log^2 n$ - many parts

$\Rightarrow O(n \log^2 n)$ for all vertices.

(lemma 54)

Insertion of a segment

Search for the first endpoint

(lemma 53)

$$O\left(\log \frac{n}{\log (\log 8/n)}\right) = O \left(\log \frac{\log (d-1)n}{\log (\log 8/n)}\right)$$

$$= O \left(\log (\log 8/n) + \log \frac{1}{\log (\log 8/n)}\right)$$

$\in O \left(\log (\log 8/n)\right)$
Follow the segment.

As before (Theorem 49) \( E(x_i) \)

\[
\frac{\text{# of new trapezoids}}{\sqrt{V(s_i)}}
\]

\( E(x_i) \in O(1) \)

\[
\sum c_{w_b} \leq \frac{n}{\log(s)} \cdot O(\log^2(n)) \in O(n)
\]

one part.

\( \log^* n \) many parts

\( \Rightarrow O(n \log^* n) \) construction cost (expected)

\( \{ \text{Expected query time } O(\log^2(n)) \} \Rightarrow 49 \)

**Theorem 57.** A simple polygon can be triangulated in expected \( O(n \log^* n) \) time.

Sketch:

**Trapezoidal map:**

\( + \) Insert edge, return a different side

\( + \) Remaining polygons

\( \Rightarrow \) Easy to triangulate

Succinctly delete, carry within