Case \( \Xi \) \( \alpha \in \beta \), \( \beta \in \hat{B}' = B \setminus a \beta \subseteq A \)

Ash by \( \Xi_2 \) \( \Rightarrow \exists F \in \Xi_2 \text{ w.l.o.g. } F \cap A = B' \)

\( \Rightarrow F, F \cup \hat{a} \beta \in \hat{\Xi} \)

(Dep. \( \Xi_2 \)) and

\( \hat{\beta} = \hat{\beta} \cup \hat{a} \beta = (F \cup \hat{a} \beta) \cap (A \cup \hat{a} \beta) \)

\( \Rightarrow A \cup \hat{a} \beta \text{ shelled by } \hat{\Xi} \text{ and claim holds} \)

\( \Rightarrow |\hat{\Xi}_2| \leq \binom{m-1}{0} + \cdots + \binom{m-1}{a-1} \)

by mod. and \( \hat{\alpha} \).

Adopting:

\( |\hat{\Xi}| = |\hat{\Xi}_1| + |\hat{\Xi}_2| \)

\[ \leq \binom{m-1}{0} + \binom{m-1}{1} + \cdots + \binom{m-1}{a-1} \]

\[ + \binom{m-1}{0} + \cdots + \binom{m-1}{a-1} \]

\[ = 1 + \binom{m}{1} + \cdots + \binom{m}{a} \]

\[ = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{a} \]

\[ \binom{m-1}{i} + \binom{m-1}{i-1} = \frac{(m-1)!}{i! (m-1-i)!} + \frac{(m-1)!}{(i-1)! (m-i)!} \]

\[ = \frac{(m-1) \cdot (m-1) + (m-1) \cdot i}{i! (m-i)!} = \frac{m}{i' (m-i)!} = \binom{m}{i} \]

\( \Rightarrow \text{ Part (i) } \checkmark \)
\( \dim_{\text{vc}} \binom{n}{d} \leq d \)

Show: \( \forall x \leq X \) \( |Y| = n \)

\( \forall \gamma \) containing at most \( \left( {m \choose 0} + \cdots + {m \choose d} \right) \) sets

Apply (i) on \( Y \) and \( \forall Y \) use \( \dim_{\text{vc}} (\gamma Y) \leq \dim_{\text{vc}} (\gamma) \leq d \)

(iii)

\( \left( {m \choose 0} + {m \choose 1} + \cdots + {m \choose d} \right) \) has no closed form, due to knapsack

Let \( \frac{1}{2} > \alpha > 0 \) (fix a value)

\[ A = \left( n + (1-\alpha)^m \right) = \sum_{i=0}^{m} \binom{m}{i} (1-\alpha)^{m-i} \]

\[ \geq (1-\alpha)^m \sum_{i=1}^{d} \binom{m}{i} \left( \frac{\alpha}{1-\alpha} \right)^i \]

\[ \geq (1-\alpha)^m \sum_{i=1}^{d} \binom{m}{i} \left( \frac{\alpha}{1-\alpha} \right)^d (\frac{\alpha}{1-\alpha})^{d-i} \]

\[ \Rightarrow \sum_{i=0}^{d} \binom{m}{i} \leq \frac{(1-\alpha)^d}{(1-\alpha)^m} \alpha ^d = S(\alpha) \]

\( S(\alpha) \) has minimum at \( \alpha = \frac{d}{m} \left( \leq \frac{1}{2} \right) \)

\[ S'(\alpha) = \frac{(1-\alpha)^d}{(1-\alpha)^m \alpha} \left( \ln(1-\alpha) - \ln(\alpha) \right) < 0 \quad \alpha < \frac{1}{2} \]
\[
\sum_{i=0}^{d} \binom{m}{i} \leq \left(1 - \frac{d}{m}\right)^{m-d} \left(\frac{m}{e}\right)^{d}
\]

as \(m \to \infty\) monotonically decreasing.

\[
\lim_{m \to \infty} \left(1 - \frac{d}{m}\right)^{m-d} \left(\frac{m}{e}\right)^{d} = e^{d}
\]

\[
\Rightarrow \frac{d}{m} \leq \frac{1}{2} \quad \sum_{i=0}^{d} \binom{m}{i} \leq e^{d} \left(\frac{m}{e}\right)^{d}
\]

Now \(d \geq \frac{m}{2}\)

\[
P(d) = \left(e^{m}\right)^{d} \quad \ln(P(d)) = d \left(1 + \ln m - \Theta_{m} d\right)
\]

\[
\ln(P(d)) > 0 \quad \text{for} \quad d \leq m
\]

\[
\left(\ln m - \ln d\right) > 0
\]

\[
\Rightarrow d \in \left[\frac{m}{2}, m\right]
\]

\[
\sum_{i=0}^{m} \binom{m}{i} \leq 2^{m} < 2^{\frac{m}{2}} \cdot e^{\frac{m}{2}} = (2e)^{\frac{m}{2}} = P\left(\frac{m}{2}\right) \leq P(d)
\]

\[\text{but} \quad 2e^{\frac{m}{2}} < 2^{\frac{m}{2}} \cdot e^{\frac{m}{2}}
\]
Proof of Theorem 44 (E-net Theorem): \( \dim_{\mathcal{C}}(\mathcal{T}) = d \leq \infty \) if \( q \geq 2 \)

Existence of E-net of size \( C \cdot d \cdot \ln r \) for

Proof works as follows:

Choose set \( N \) of size \( s = C \cdot d \cdot \ln r \) (also \( f \cdot C \))

\( \mu \)-randomly sample \( X \) from \( S \) with independent random draws

\( \Rightarrow \mu \)-probability that \( N \) is a \( \frac{1}{r} \)-net is

greater than \( \Theta \). (So that has to be an \( \mu \) magic)

Proof \( \Rightarrow \) Existence, but also a bound for \( C \)

Preliminaries:

- Proof site, randomized algorithm for finding \( \frac{1}{r} \)-net

- \( \mu \)-random sampling

\( \Rightarrow \mu \)-probability that \( S \) has some point is \( E[Y] = \mu(Y) \)

- \( S \) independent random draws, “lay back”

Some of size \( s \) is not a set!

Technical lemma distribution

Lemma 46: Let \( X = X_1 + X_2 + \ldots + X_n \) where \( X_i \)

are independent random variables

\[ \Pr(\frac{1}{2} \geq X_i) \geq \frac{1}{2} \text{ for } np > 8 \]

Proof: By Chebyshev’s Inequality

\[ \Pr(|X - E(X)| > \alpha) \leq \frac{V(X)}{\alpha^2} \]

Derivation of Chebyshev’s inequality. Value of random variable in a region around the expected value.

\[ V(X) = E((x - E(x))^2) \]
$E(x) = n \cdot p$

$V(x) = \sum_{i=1}^{n} V(x_i)$

$\leq \sum_{i=1}^{n} E((x_i - E(x_i))^2) \leq np$

$E((x_i - p)^2) = E(x_i^2 - 2px_i + p^2)$

$= E(x_i^2) - 2px_i + p^2$

$= p - p^2 \leq p$

$E(x_i^2) = (1-p)0^2 + p \cdot 1^2 = p$

$E(x) = (1-p)0 + p \cdot 1 = p$

$P(\frac{X - np}{\sqrt{np}} < \frac{a}{2}) \leq P(|X - E(x)| > \frac{np}{2}) \leq \frac{np}{(np + 2)} = \frac{4}{np}$

$P(\frac{X > np}{2}) \geq 1 - \frac{4}{np} > \frac{1}{2}$ for $np \geq 8$. \(\square\)
Now proof continued.

w.l.o.g. \( M(F) \geq \frac{a}{F} \vee \forall F \in \mathcal{T} \) (otherwise leave them out)

Two independent random chains \( N, M \) of size \( s \) ("layback" set)

Ends:

\[ E_0: \ N \text{ is not an } \frac{a}{F} \text{-net}, \ i.e. \ \exists S \in \mathcal{T} : S \cap N = \emptyset \]

\[ E_1: \ \exists S \in \mathcal{T} : S \cap N = \emptyset \text{ and } |S \cap M| > \epsilon \]

\( ( \text{Bound } \Pr(E_0) \text{ from above by } \Pr(E_1), \Pr(E_0) \leq 1) \)

\[ \text{Claim } \Sigma \Pr(E_1) \leq \Pr(E_0) \quad \text{clear } E_1 \text{ requires more detail} \]

\[ \text{Claim } \Pi \Pr(E_0) \leq 2 \Pr(E_1) \quad (E_0 \Rightarrow E_1) \]

Proof (Claim \( \Pi \)): Conditional Probability

\[ \Pr(E_1|N) \quad \text{for fixed } N, \ M \text{ is random} \]

if \( N \text{ is } \frac{a}{F} \text{-net} \Rightarrow \neg E_0 \wedge \neg E_1 \Rightarrow \Pr(E_1|N) = 0 \)

\[ \Pr(E_0|N) = 0 \]

if \( N \text{ is not an } \frac{a}{F} \text{-net} \)

\[ \Rightarrow \exists S_N \in \mathcal{T} \text{ with } N \cap S_N = \emptyset \]

\[ \Rightarrow \Pr(E_0|N) = 1 \]

\[ \Rightarrow \Pr(E_1|N) \geq \Pr(|S_N \cap M| > \epsilon) \geq \ldots \]

\( ( \text{some other set } S' \text{ might also} \)

\( \text{fulfill } S' \cap N = \emptyset \text{ and } |S' \cap M| > \epsilon) \)
\[ x = \frac{S}{2r}, \quad M(S_N) \geq \frac{1}{r} \]

\[ \Rightarrow x = \frac{S}{2r} \leq \frac{SM(S_N)}{z} \]

\[ \vdash \quad \text{Prob} (|S_N \cap M| \geq \frac{SM(S_N)}{2}) \geq \frac{1}{2} \]

\[ \left| |S_N \cap M| \text{ random variable} \right| \]

\[ \sum_{i=1}^{m} X_i, \text{ drawing of } M \]

\[ \uparrow \quad \text{independent ("lay back" is important rule)} \]

\[ x_i = 1 \Rightarrow x \in S_N \text{ draw } \]

\[ \text{Probability: } x \in S_N \text{ is } M(S_N) \]

\[ S \text{-draws } \Rightarrow \text{ Lemma 46 applicable} \]

\[ (SM(S_N) \geq 8, \quad \frac{S}{r} \geq 8) \]

\[ \text{Algorithm: } \forall N \quad \text{Prob} (E_0 | N) \leq 2 \text{ Prob} (E_1 | N) \]

\[ \text{for all fixed } N \]

\[ \Rightarrow \quad \text{Prob} (E_0) \leq 2 \text{ Prob}(E_1) \]

\[ \sum_{i} P(A) = \sum_{i} P(B_i) \cdot P(A|B_i) \]

\[ \sum_{i} P(B_i) = \text{total probability from partial probability, } \]

\[ \mathbb{U} B, \quad \text{total probability from partial probability, } \]
Now differ on bound for $\Prib(E_1)$ (reps $\Prob(E_0)$ small)

There's a different chance of $N$ and $M$.

- Choose a semi $A$ of $2$ elements of $X$ ("play back", $\Prob(A)$)
- Choose (randomly) $S$ positions for $N$ to remain on, build $M$

$N$, $M$ are chosen with the same probability as before $V$

Let $A$ be fixed. Consider $\forall x \in S \in \mathcal{T}$ ($N$, $M$ randomly)

Let $P_s = P(\left(N \cap S = \emptyset \text{ and } |M \cap S| \geq \lambda \right) \mid A)$

Claim III: $P_s \leq \frac{-cd}{q} = \Prob\left(E_1 \mid A\right)$ small $V$

Proof:

Case 1 $|A \cap S| < \lambda \Rightarrow |M \cap S| < \lambda \Rightarrow P_s = 0$

Case 2 $|A \cap S| \geq \lambda$

$= \Pr(\left(N \cap S = \emptyset \mid A\right)$

only one point

$= \Pr\left(T_e \text{ positions for } N \text{ out of } A \right)$

Avoid $\left(\text{at most } \lambda \right)$ positions of

$S$-elements in $A$

"Positions $V"
\[
\left( \frac{2s-x}{s} \right) = \frac{(2s-x)!}{(s-x)!} \frac{s!}{(2s)!}
\]

\# possible pairs \( \frac{s!}{(2s)!} \)

\# possible different positions \( \text{of } s \text{ out of } 2s \text{ positions} \)

\[= \frac{(s-9)(s-8)(s-7) \ldots (2s-9)(2s-8)}{(s-1)(s-2) \ldots (2s)}\]

\[\leq \left( \frac{2s-x}{2s} \right)^s = \left( 1 - \frac{x}{2s} \right)^s = \left( 1 - \frac{x}{s} \right)^s \leq e^{-\frac{x^2}{2}}\]

\[\Rightarrow e^{-\frac{Cd}{4}} = \frac{Cd}{4}\]

\[x = \frac{s}{2r}, \ s = \text{col} \text{or} \text{row}\]

\[\text{Ann} \ (\bar{v}) \leq d \text{ was not used up do now}\]

Sh. Func. Lemma

\[|A| \leq 2s \Rightarrow \exists A \text{ span at most } \left( \frac{2es}{d} \right)\]

\text{affected elements.}
Let $A$ be fixed but $S$ be a variable

\[ E(s) := (S \cap N = \emptyset \text{ and } |S \cap M| > x) \]

depends on $A, N, S$ as $P_s$

\[ P[E, IA] \leq \sum_{\text{pairwise disjoint}} P[E(s)] \]

"$s \in \tau : E(s)$"

\[ \leq \max_s P_s \]

\[ \leq (2es)^d \cdot C_d \]

\[ \leq \left( 2e (\log r \cdot r - \frac{C}{d}) \right) \leq \frac{1}{2} \]

for $d, r > \frac{1}{2}$ and sufficient large $C$.

Algorithm: $\text{Prob}(E_0) \leq 2 \text{Prob}(E_i) \leq 2 \sum_{A} \text{Prob}(E, IA) \text{Prob}(A) \leq \frac{1}{2} \sum_{A} \leq 1$

Probability that a randomly chosen $N$ is at most an $\frac{1}{r}$th

is smaller than $1$ $\Rightarrow$ Probability that a randomly choose $N$

is at an $\frac{1}{r}$th $\Rightarrow$ There exists such a net $N$. 
Conclusions:

Art gallery: \( \text{dim}_{VC}(\mathbb{F}) \leq 14 \)

\( \mathbb{F} = \frac{9}{2} \text{vis}_p(e) | e \in P \)

\( \text{vis}_p(e) \geq \frac{1}{10} \text{vol}(P) \quad \tau = 10 \)

\( \Rightarrow d = 14 \quad \tau = 10 \)

C so that \( \left( 2e \cos 10 \ln 10 - 10 - \frac{c}{0.14} \right) \leq \frac{1}{2} \)

\( C = 13.08 \)

\( S = 13.08 \times 14 \times 10 \ln 10 = 4218 \) guards

(Art Gallery) Theorem 4.7 There is a constant \( D > 0 \)

So too! For any area \( P \) bounded by a simple closed Jordan curve and any point \( P \) sees at least \( \frac{1}{\tau} \) of the volume of \( P \) the number of guards required is smaller than \( D \).