Computing a smallest transversal can be difficult.

Example 1:

\[ R = \text{finite set of (axis-parallel) rectangles in } \mathbb{R}^2 \]

\[ P = \text{finite set of points} \]

**Problem:** Find minimal subset \( P_{\text{min}} \subseteq P \) so that any \( r \in R \) contains a point in \( P_{\text{min}} \).

**NP-complete** (also for axis-parallel lines \( v \))

**Try:** \( X = P, \quad T = \{ P \cap r \mid r \in R \} \)

Example 2:

\[ R = \text{finite sets of intervals along } X\text{-axis} \]

\[ P = \text{finite set of points} \]

**Problem:** Find minimal subset \( P_{\text{min}} \subseteq P \) so that any \( r \in R \) contains at least one point.
Does always \( \nu(t) = \tilde{\tau}(t) \) hold?

No, example \( \nu < \tau \)

\[ X = \mathbb{R}^2, \quad \tilde{\tau} = \sum_{n} \text{lines in general position} \]

\[ \nu(t) = 1 \]

\[ \tilde{\tau}(t) = \left( \frac{n^2}{2} \right) \]

\((\nu > \tau \text{ not in general})\)

**Proposition:** \( \nu(t) \leq \tilde{\tau}(t) \)

**Proof:** For \( m \) disjoint sets of \( \tilde{\tau} \) at least \( m \) stabbing points are necessary.

Relaxation of the transversal \( \Rightarrow \) \&-net

Large sets should be easier hit by a transversal than small ones!
Simple, special case $X$ is finite.

Size of a set by cardinality.

**Definition 48** (ε-net, special case)

Let $(X, \tau)$ be a set system on $\varepsilon \in [0, 1]$ a real number. A set $N \subseteq X$ (not necessarily $N \subseteq \tau$) is called an ε-net for $(X, \tau)$ if $N \cap S \neq \emptyset$ for all $S \in \tau$ with $|S| \geq \varepsilon |X|$. 

- ε-net is transversal for all sets larger than $\varepsilon |X|$

- Convenient to write $\varepsilon = \frac{1}{r}$ for $r > 1$ \( r \in \mathbb{R} \).

- Epsilon net theorem \( \Rightarrow \) condition on $\tau$:
  - existence of $\frac{1}{r}$ nets of size $O(r \log r)$

- Art gallery: all polygons you can see from two \( \frac{1}{r} \) vol($P$) of
  - # guards is $O(r \log r)$.

\( X = P \) not finite

General definition, $\mu$ a probability measure.

In most cases $\mu$ is volume

or $\mu$ gives cardinality $V$.

\( ({}^{\text{Pint, sub}}) \)
$P \text{ polytope:}$

$$M(P) = 1 \quad M(\text{vis}_P(p)) = \frac{\text{vol}(\text{vis}_P(p))}{\text{vol}(P)} \quad \forall \varepsilon = \frac{1}{p}$$

Definition 4.3 (ε-net)

$N \subseteq X$ is called ε-net for $(X, \mathcal{T})$

$$\iff \forall F \in \mathcal{T} \quad (M(F) \geq \varepsilon \implies F \cap N \neq \emptyset).$$

In other words: $N$ is a transversal of a sub system of $\mathcal{T}$ with sufficiently large sets.

Fits to our and Galley question: $M(\text{vis}_P(p)) \geq \frac{1}{p} M(P)$

$\forall p \in P$

Seeking for sufficient conditions for the existence of finite (small) ε-nets. VC-dim

Example 1

$X = \text{unit square} \quad \mathcal{G} = \{0, 1\}^2$

$\mathcal{T} = \{ \mathcal{F} \subseteq \mathcal{G} \text{ area of single poly} \}$

No finite ε-net exists.

Assume finite net $\mathcal{N}$ with

Fails on always polytope volume arbitrarily close to $1$ but do not contain the net point.
Example 2 \( X = \mathbb{Q} \) unit square

\[ \mathcal{T} = \{ F \subset \mathbb{Q} \text{ square} \} \]

For all \( \varepsilon > 0 \) there is a point \( c \)-net (depending on \( \varepsilon \))

Grid with distance \( \delta \)

\( \varepsilon \)-net for \( (x, y) \) with \( \varepsilon = 2\delta^2 \)

Unit square partitioned between grid points

Maximal size

Rectangle \( \sqrt{2} \cdot d \) side length.

Theorem 44 (\( \varepsilon \)-net theorem): Let \( (X, \mathcal{F}) \) be

a set system with measure \( \mu \) let \( \text{dim}_{\text{VC}}(\mathcal{F}) = d < \infty \).

For \( r > 2 \) there is an \( \varepsilon \)-net for \( \mathcal{T} \) with

size at most \( C \cdot d \cdot r^d \), where \( C \) is an independent constant.

Bound on \( C \): Fundamental lemma bounding

the number of distinct sets in a system of given VC dimension.
Definition

Let \((X, \mathcal{F})\) be a set system \(\mathcal{T} \subseteq \mathcal{P}(X)\)

\[
\bar{N}_\mathcal{F}(m) := \max_{\substack{\mathcal{T} \subseteq \mathcal{P}(X) \\
Y \subseteq X \\
|Y| = m}} |\mathcal{T} \cap Y|
\]

is denoted as the \textit{shackle function} of \((X, \mathcal{F})\).

\(\bar{N}_\mathcal{F}(m)\) maximum possible number of distinct
intersections of the sets \(\mathcal{T}\) with an \(m\)-point subset \(Y \subseteq \mathcal{T}\) is not required, and also that \(Y\) is fully shelled.

\[
\mathcal{T} \cap Y = \{ E \cap Y | E \in \mathcal{T} \}
\]

Relationship between \(\bar{N}_\mathcal{F}(m)\) and \(\text{dim}_{\text{VC}}(\mathcal{F})\)

\textbf{Claim} \quad \text{dim}_{\text{VC}}(\mathcal{F}) = 0 \quad \Leftrightarrow \quad \forall m \quad \bar{N}_\mathcal{F}(m) = 2^m

(any subset intersected with different sets)

\(\leq\) any \(m\) set is fully shelled \(\Rightarrow\) \text{dim}_{\text{VC}}(\mathcal{F}) = 0 \Rightarrow \bar{N}_\mathcal{F}(m) = 2^m

\(\Rightarrow\) \(\exists t \leq m \quad |\mathcal{T}| = 2^m \quad \text{exists} \quad \text{dim}_{\text{VC}}(\mathcal{F}) = 0\)

Set \(A \subseteq B \quad |A| = m \quad |A| = m \quad |B| = 2^m \quad \Rightarrow \quad \bar{N}_\mathcal{F}(m) = 2^m \quad \text{for all } m.\)
Lemma 46, (Shelah Frodon Lemma) \( \times \) finite

(i) Let \(|X| = m\) and \(\dim_{VC}(\mathcal{F}) \leq d\).

Then \(\mathcal{F}\) consists of at most

\[
\binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{d}
\]

subsets of \(X\).

(ii) For \(\dim_{VC}(\mathcal{F}) \leq d\)

\[
\overline{t}_{\mathcal{F}}(m) \leq \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{d}.
\]

(iii) \(\binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{d} \leq \left(\frac{e}{e-d}\right)^d e^{O(m^d)}\)

Proof:

(\(\forall, n\)) \((X, \mathcal{F})\) and \(X' \subseteq X\)

\[
\Rightarrow \dim_{VC}(\mathcal{F} | X') \leq \dim_{VC}(\mathcal{F})
\]

Let \(C \subseteq X'\); assume \(C\) is shattered by \(\mathcal{F} | X'\)

\[
\Rightarrow \forall B \subseteq C \exists F' \in \mathcal{F} \text{ such that } B = C \cap F'
\]

\[
\Rightarrow \forall B \subseteq C \exists F \in \mathcal{F} \quad B = C \cap (F \cap X') \subseteq C \cap \mathcal{F}
\]

\[
\Rightarrow C \text{ is shattered by } \mathcal{F}
\]
Part (i). Induction on \( d \), nested induction on \( m \)

\[ d \leq m \text{ clear no more than } |X| \text{ elements.} \]

**Ind. base** \( d = 0 \) arbitrary \( m \)

\[ \dim_{vc}(\varnothing) = 0 \implies \text{no set of one element is shattered} \]

\[ \Rightarrow a \in X,  \exists \varnothing, \exists a \varnothing \subseteq \varnothing \varnothing \]

\[ \text{either no } F \subseteq \varnothing \text{ or } F \cup \varnothing \subseteq \varnothing \varnothing \text{ or } F \varnothing \subseteq \varnothing \varnothing \checkmark \]

\[ \Rightarrow F \varnothing \subseteq \varnothing \varnothing \Rightarrow \forall F \forall a \Rightarrow F = \varnothing \varnothing \]

\[ \Rightarrow |\varnothing| = 1 = \binom{m}{0} = \binom{m}{d} \]

**Ind. step** \( d \geq 1 \) (also \( m \geq 1 \)) let \( a \in X \)

\[ X_1 = X \setminus \{a\} \quad \varnothing_1 = \varnothing \setminus \{a\} \Rightarrow \{F \cap X_1 \mid \varnothing \subseteq F \subseteq \varnothing_1\} \]

\[ \dim_{vc}(\varnothing_1) \leq \dim_{vc}(\varnothing) \leq d \]

Induction on \( m \)

\[ |\varnothing_1| \leq \binom{m-1}{0} + \binom{m-1}{1} + \cdots + \binom{m-1}{d} \]

How many sets can \( \varnothing \) have more than \( \varnothing_1 \)?

Consider the mapping

\[ f : \varnothing \rightarrow \varnothing_1 \]

\[ A \rightarrow A \cup \{a\} \]
We have

\[ p(A_1) = p(A_2) \quad \iff \quad A_2 = A_1 \cup \exists a? \]

or

\[ A_1 = A_2 \cup \exists a? \]

Let \( \tilde{t}_2 = \{ \alpha \in T \mid \alpha \notin A \text{ and } A \cup \exists a? \in T \} \) set syst on \( X_4 \).

For some element of \( A_1, A_2 \) with the smallest number of elements

\[ |\tilde{t}_1| = |\tilde{t}_1| + |\tilde{t}_2| \]

because

\[ |\tilde{t}_2| \text{ numb of set with } a^* \text{ or wknod } a^* \quad (\text{ has counterpart}) \]

plus

\[ |\tilde{t}_1| \text{ counterparts to } \tilde{t}_2 \text{ with } a \in A' \]

and sets in \( T \) with \( F \cap X_4 = F \).

Claim:

\[ \dim \psi (\tilde{t}_2) \leq d-1 \quad \text{set syst on } X_4 \]

Proof: Show: \( \forall A \subseteq X_4 \) A shald by \( \tilde{t}_2 \Rightarrow A \cup \exists a? \)

Shald by \( \tilde{t} \)

If this is true we have

\[ |A| \leq |A|_{\tilde{t}} = |A \cup \exists a?| \leq \dim \psi (\tilde{t}) \leq d \]

\[ \implies |A| \leq d-1 \]

Now let \( A \subseteq X_4, B \subseteq A \cup \exists a? \)

Case 1. \( a \notin B \):

\( B \subseteq A \Rightarrow \exists F \in \tilde{t}_2 \text{ with } F \cap A = B \)

(Ash by \( \tilde{t}_2 \))

\[ \implies F \cap A \cup \exists a? = B \]
\( f_n = x_1 \times \sum x \) 

\( x_1 = x \times \sum x \) \quad \{x, |x| \leq m \text{ and } \text{np} \}

\( f_2 = \left\{ A \in T \mid A \cap A \text{ and } A \cup x \in T \right\} \)

\( T = \left[ f_1 \right] + [f_2] \quad \text{and} \quad \omega_{C_1}(T) \leq \alpha \cdot n \text{ w.r.t. np} \)