6. Construction of AVD

Finite Part of AVD

- Let $\Gamma$ be a simple closed curve such that all intersections between bisecting curve lie inside the inner domain of $\Gamma$
- Consider a site $\infty$, define $J(p, \infty) = J(\infty, p)$ to be $\Gamma$ for all sites $p \in S$, and $D(\infty, p)$ to be the outer domain of $\Gamma$ for all sites $p \in S$.

Incremental Construction

- Let $s_1, s_2, \ldots, s_n$ be a random sequence of $S$
- Let $R_i = \{\infty, s_1, s_2, \ldots, s_i\}$
- Iteratively construct $V(R_2), V(R_3), \ldots, V(R_n)$

General Position Assumption

- No $J(p, q)$, $J(p, r)$ and $J(p, t)$ intersect the same point for any four distinct sites, $p, q, r, t \in S$
  $\rightarrow$ Degree of a Voronoi vertex is 3

Remark

- For $1 \leq i \leq n$ and for all sites $p \in R_i$, $VR(p, R_i)$ is simply connected, i.e., path connected and no hole
- If $J(p, q)$ and $J(p, r)$ intersect at a point $x$, $J(q, r)$ must pass through $x$
Basic Operations

• Given \( J(p, q) \) and a point \( v \), determine \( v \in D(p, q) \), \( v \in J(p, q) \), or \( v \in D(q, p) \)

• Given a point \( v \) in common to three bisecting curves, determine the clockwise order of the curves around \( v \)

• Given points \( u \in J(p, q) \) and \( w \in J(p, r) \) and orientation of these curves, determine the first point of \( J(p, r) \mid_{(w, \infty]} \) crossed by \( J(p, q) \mid_{(v, \infty]} \)

• Given \( J(p, q) \) with an orientation and points \( v, w, x \) on \( J(p, q) \), determine if \( v \) come before \( w \) on \( J(p, q) \mid_{(x, \infty]} \)

Notation: Give a connected subset \( A \) of \( \mathbb{R}^2 \), \( \text{int}A \), \( \text{bd}A \), and \( \text{cl}A \) mean the interior, the boundary, and the closure of \( A \), respectively.

Conflict Graph \( G(R) \), where \( R \) is \( R_i \) for \( 2 \leq i \leq n \)

• bipartitle graph \((U, V, E)\)

• \( U \): Voronoi edges of \( V(R) \)

• \( V \): Sites in \( S \setminus R \)

• \( E \) : \( \{(e, s) \mid e \in V(R), s \in S \setminus R, e \cap VR(s, R \cup \{s\}) \neq \emptyset\} \)

\( - \) a conflict relation between \( e \) and \( s \).

Remark:
a Voronoi edge is defined by 4 sites under the general position assumption
Lemma 1
Let $R \subseteq S$ and $t \in S \setminus R$. Let $e$ be the Voronoi edge between \(\text{VR}(p, R)\) and \(\text{VR}(q, R)\). \(e \cap \text{VR}(t, R \cup \{t\}) = e \cap R(t, \{p, q, r\})\). (Local Test is enough)

**Proof:**

\(\subseteq\): Immediately from \(\text{VR}(t, R \cup \{t\}) \subseteq \text{VR}(t, \{p, q, t\})\)

\(\supseteq\): Let \(x \in e \cap \text{VR}(t, \{p, q, t\})\)

- Since \(x \in e\), \(x \in \text{VR}(p, R) \cup \text{VR}(q, R)\) and \(x \notin \text{VR}(r, R) \supseteq \text{VR}(r, R \cup \{t\})\) for any \(r \in R \setminus \{p, q\}\).
- Since \(x \in \text{VR}(t, \{p, q, t\})\), \(x \notin \text{VR}(p, \{p, q, t\}) \cup \text{VR}(q, \{p, q, t\}) \supseteq \text{VR}(p, R \cup \{t\}) \cup \text{VR}(q, R \cup \{t\})\)
- \(x \notin \text{VR}(r, R \cup \{t\})\) for any site \(r \in R \rightarrow x \in \text{VR}(t, R \cup \{t\})\)

Insertiong \(s \in S \setminus R\) to compute \(V(R \cup \{s\})\) and \(G(R \cup \{s\})\) from \(V(R)\) and \(G(R)\). Handle a conflict between \(s\) and a Voronoi edge \(e\) of \(\text{VR}(R)\)

Lemma 2
\(\text{cl} e \cap \text{cl} \text{VR}(s, R \cup \{s\}) \neq \emptyset\) implies \(e \cap \text{VR}(s, R \cup \{s\}) = \emptyset\)

**Proof**
- Let \(x\) belong to \(\text{cl} e \cap \text{cl} \text{VR}(s, R \cup \{s\})\)
- \(x\) is an endpoint of \(e\):
  - \(x\) is the intersection among three curves in \(R\)
  - For any \(r \in R\), \(J(s, r)\) cannot pass through \(x\) due to the general position assumption
  - \(x \in D(s, r) \rightarrow \text{the neighborhood of } x \in D(s, r)\)
  - \(\exists y \in e\) belongs to \(\text{VR}(s, R \cup \{s\})\)
- \(x \in e \cap \text{bd} \text{VR}(s, R \cup \{s\})\)
  - \(x \in J(p, q) \cap J(s, r)\)
  - a point \(y \in e\) in the neighborhood of \(x\) such that \(y \in \text{VR}(s, R \cup \{s\})\)
Theorem 2

$V(S)$ can be computed in $O(n \log n)$ expected time

- $\sum_{3 \leq i \leq n} O(\sum_{(e,s) \in G(R_{i-1})} \deg_{G(R_{i-1})}(e))$
- Let $e$ be a Voronoi edge of $V(R_i)$ and let $s$ be a site in $S \setminus R_i$ which conflicts $e$.
- The conflict relation $(e, s)$ will be counted only once since the counting only occurred when $e$ is removed
  - Let $s_j$ be the earliest site in the sequence which conflicts with $e$. Then $(e, s)$ will be counted in $\deg_{G(R_{j-1})}(e)$
- Time proportional to the number of conflict relations between Voronoi edges in $\bigcup_{2 \leq i \leq n} V(R_i)$ and sites in $S$
- The expected size of conflict history is $-C_n + \sum_{2 \leq i \leq n}(n - j + 1)p_j$
  - $C_n$ is the expected size of $\bigcup_{2 \leq i \leq n} V(R_i)$
  - $p_j$ is the expected number of Voronoi edges defined by the same two sites in $V(R_j)$
- Since $C_n = O(n)$ and $p_j = O(1/j)$, the expected run time is $O(n \log n)$