Online Motion Planning MA-INF 1314
Window Shopper

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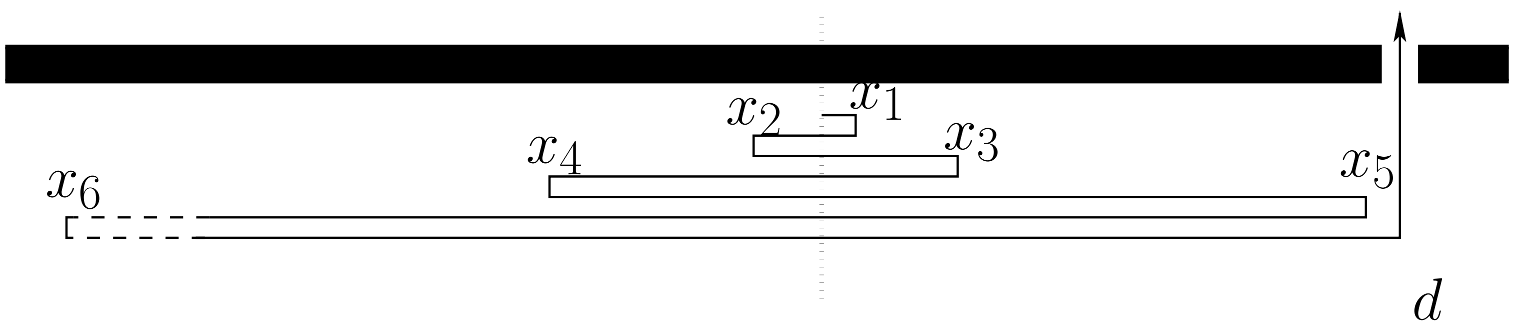
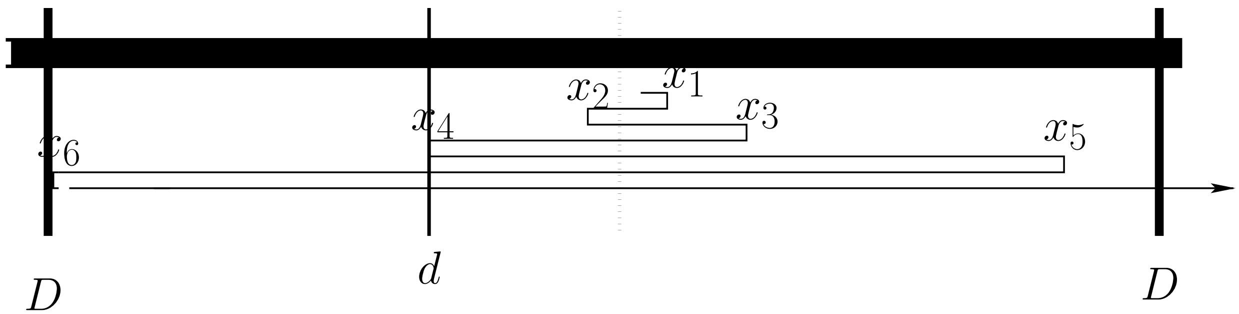
Rep. 2-ray search: Optimality for equations!

- Set: $\frac{\sum_{i=1}^{k+1} x'_i}{x'_k} = \frac{(C-1)}{2}$ for all k
- $\sum_{i=1}^{k+1} x'_i - \sum_{i=1}^{k} x'_i = \frac{(C-1)}{2}\left(x'_k - x'_{k-1}\right)$
- Thus: $C'\left(x'_k - x'_{k-1}\right) = x'_{k+1}$, Recurrence!
- Solve a recurrence! Analytically! Blackboard!
- Characteristical polynom: No solution $C' < 4$
- $x'_i = (i+1)2^i$ with $C' = 4$ is a solution! Blackboard! Optimal!



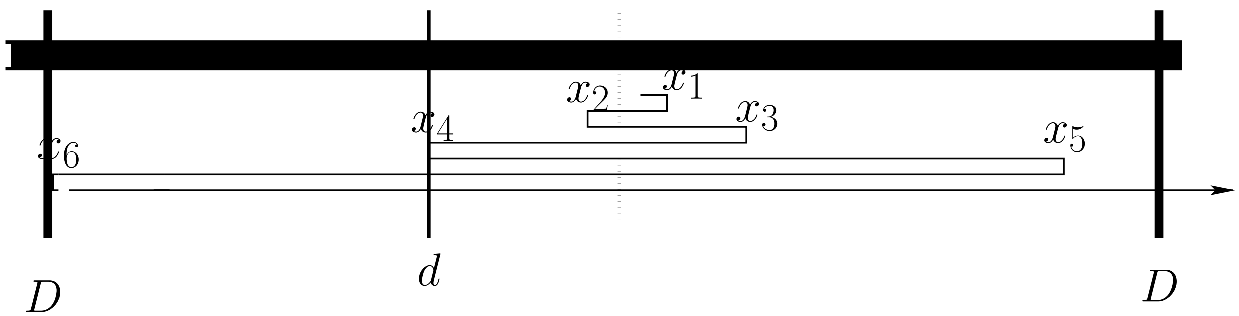
Rep.: 2-ray search, restricted distance

- Assume goal is no more than dist. $\leq D$ away
- Exactly D! Simple ratio 3!
- Find optimal startegy, minimize C!
- Vice-versa: C is given! Find the largest distance D (reach R) that still allows C competitive search.
- One side with $f_{\text{Ende}} = R$, the other side arbitrarily large!



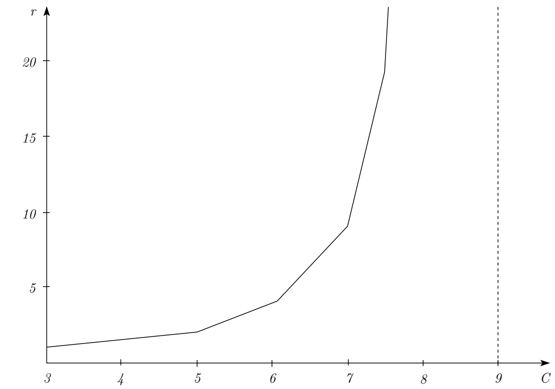
Rep.: 2-ray search, maximal reach R

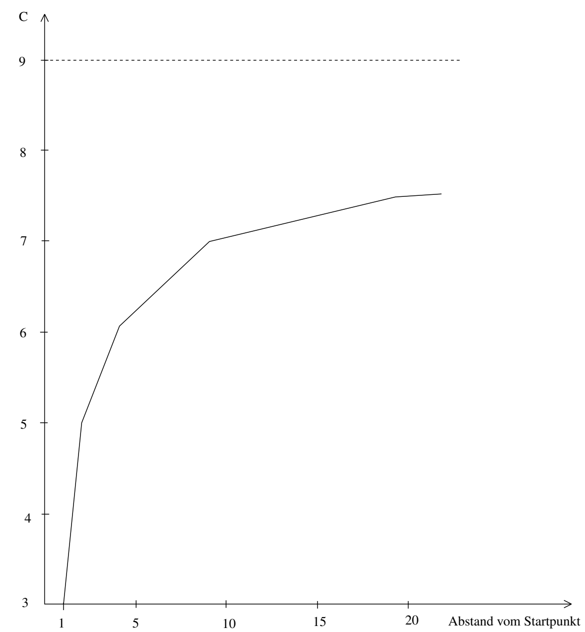
- C given, optimal reach R!
- **Theorem** The strategy with equality in any step maximizes the reach R !
- Strategy: $\frac{\sum_{i=1}^{k+1} x_i}{x_k} = \frac{(C-1)}{2}$, first step: $x_1 = \frac{(C-1)}{2}$
- Recurrence: $x_0 = 1$, $x_{-1} = 0$, $x_{k+1} = \frac{(C-1)}{2}(x_k - x_{k-1})$
- Strategy is optimal! By means of the Comp. Geom. lecture!



Rep: Solutions!

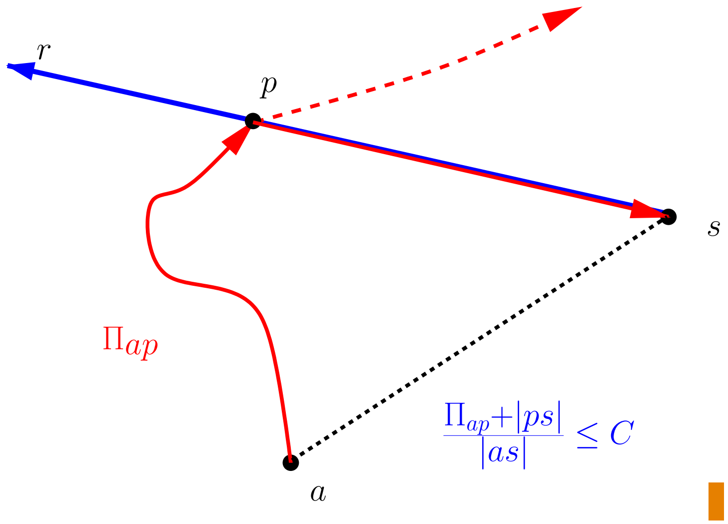
- $f(C) :=$ max. reach depending on C
- Vice versa, R given, binary search





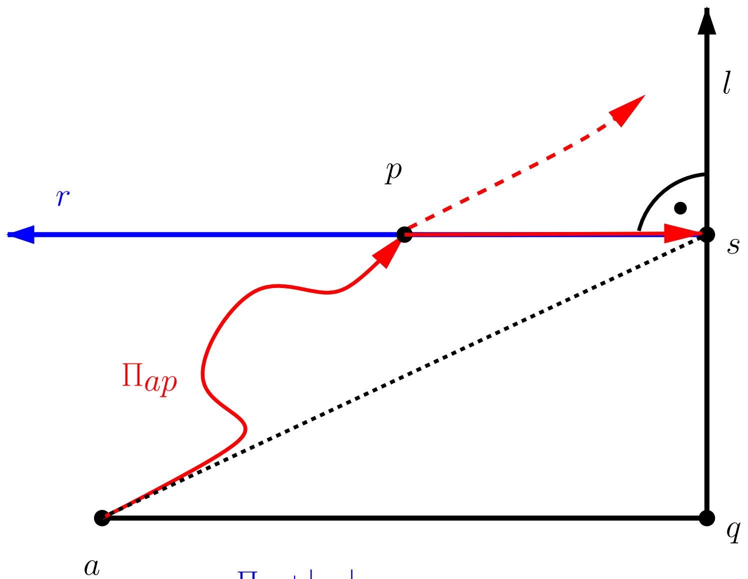
WH: Searching for the origin of ray

- Unknown ray r in the plane, unknown origin s Startpoint a
- Searchpath Π, hits r, detects s, move to s
- Shortest path OPT, build the ratio
- Π has *competitive ratio* C if inequality holds for all rays
- Task: Find searchpath Π with the minimal C
- Special Problem: Window Shopper



WH: The Window-Shopper-Problem

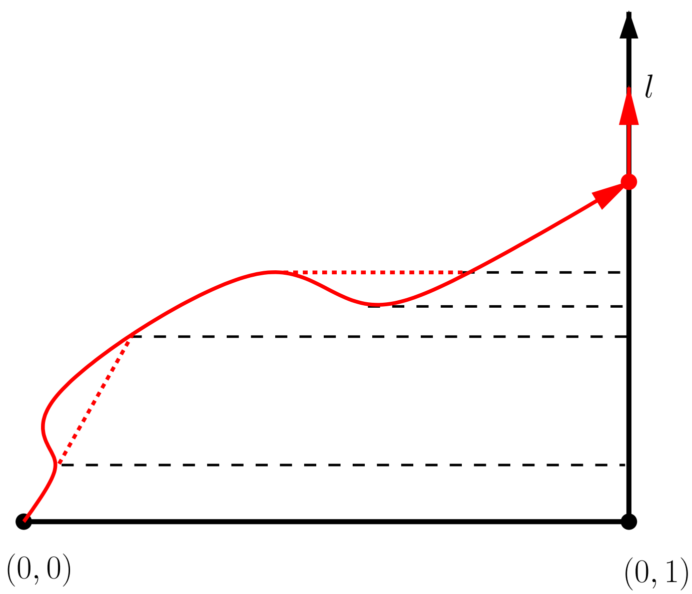
- Unknown ray starts at s on *known* vertical line l(window)
- Ray starts perpendicular to l
- aq runs parallel to r
- *Motivation:* Move along a window until you *detect* an item
- Move to the item



$$\frac{\Pi_{ap}+|ps|}{|as|} \leq C$$

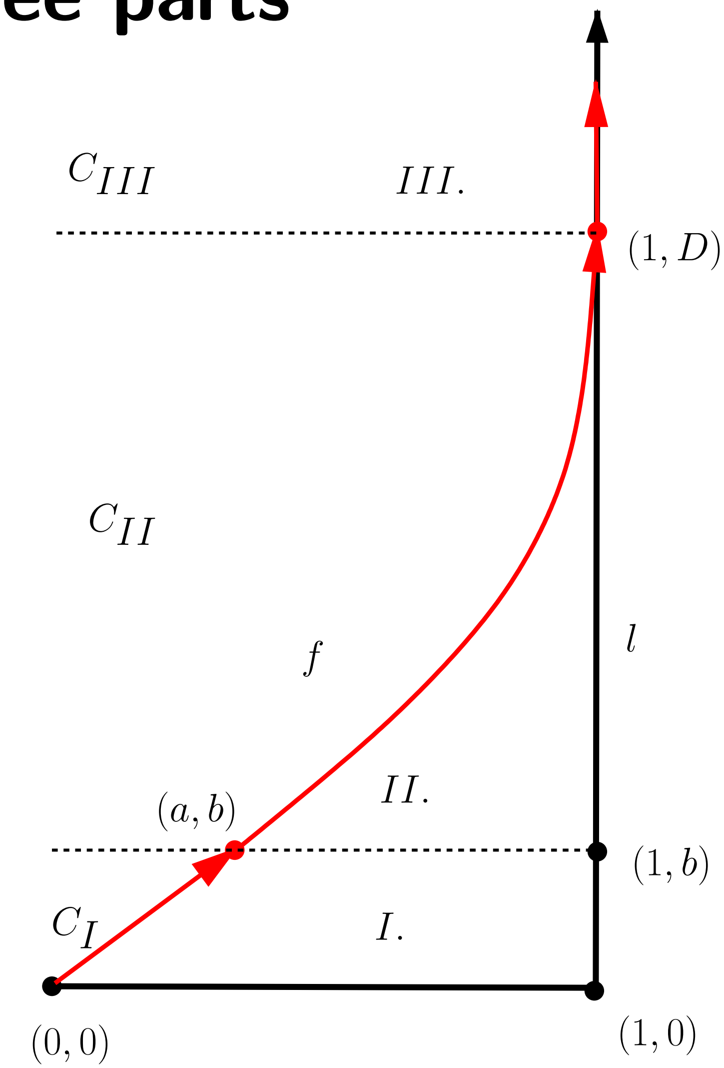
WH: Some observations

- Any reasonable strategy is monotone in x and y
- Otherwise: Optimize for some s on l
- Finally hits the *window*
- Ratio is close to 1 in the beginning, but bigger than 1
- Ratio goes to 1 at the end



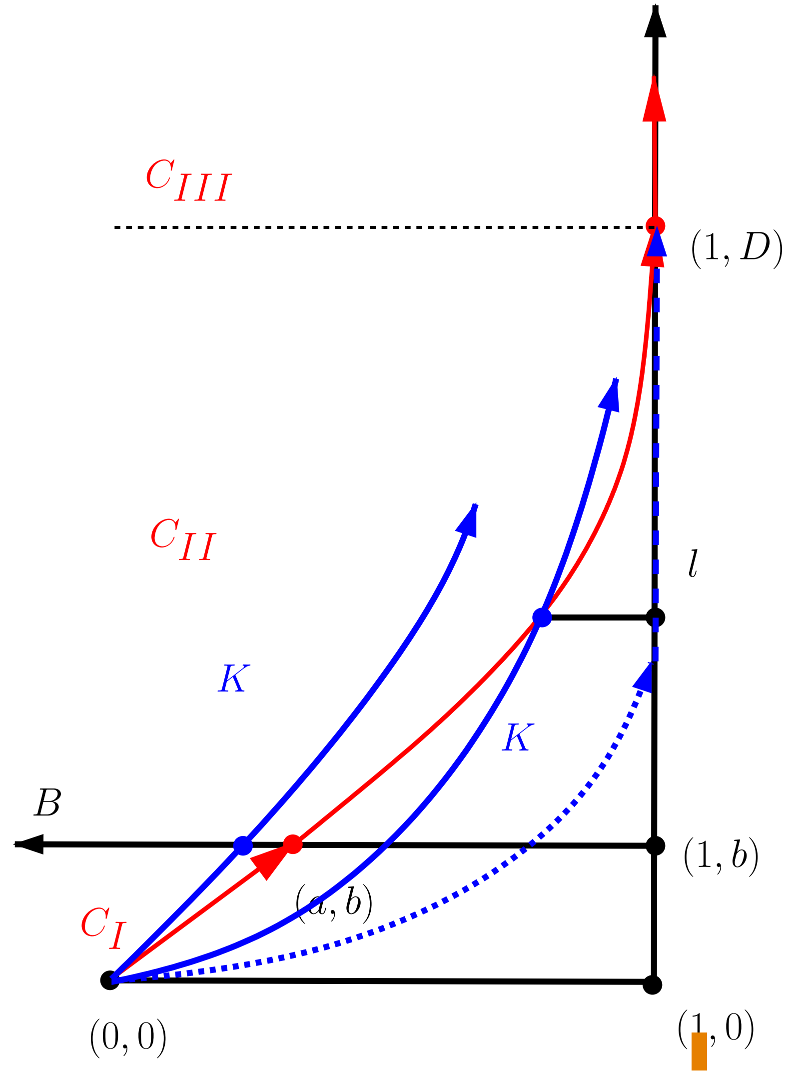
Strategy design: Three parts

- A line segment from $(0,0)$ to (a,b) with increasing ratio for s between $(1,0)$ and $(1,b)$
- A curve f from (a,b) to some point $(1,D)$ on l which has the same ratio for s between $(1,b)$ and $(1,D)$
- A ray along the *window* starting at $(1,D)$ with decreasing ratio for s beyond $(1,D)$ to infinity
- Worst-case ratio is attained for all s between $(1,b)$ and $(1,D)$



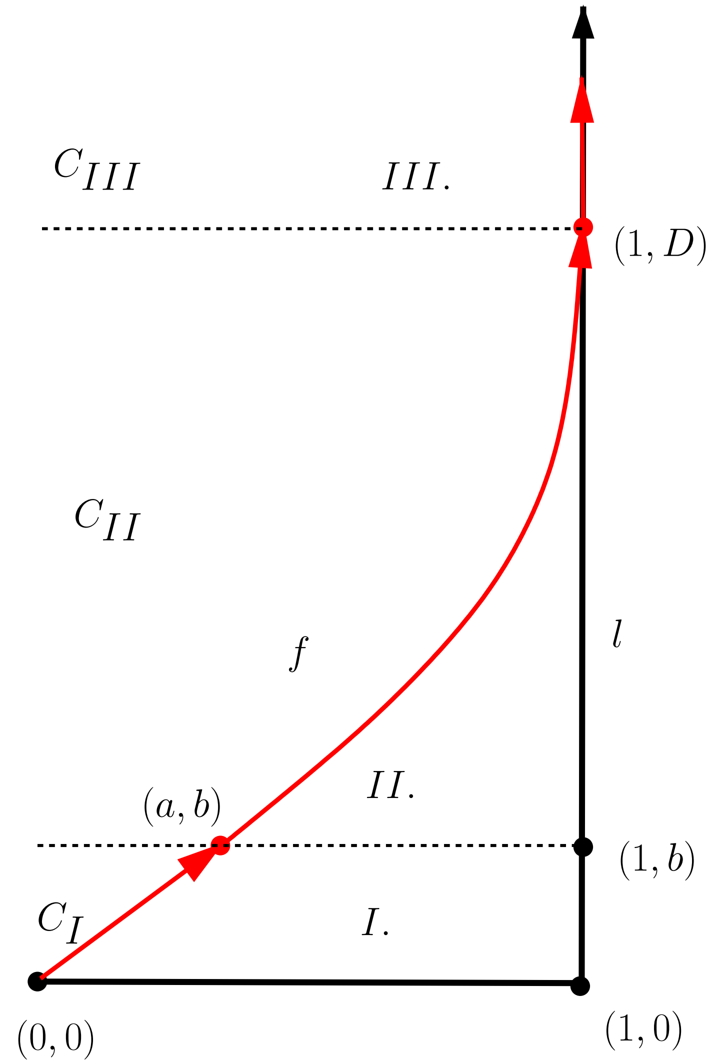
Optimality of this strategy

- By construction
- Curve has the given property
- Proof: Curve is convex
- Assume: Optimal curve K
- K hits ray B at some point (x, b)
- Two cases:
  - Hits B to the left of a: ratio is bigger
  - Cross f beyond B from the right: ratio is bigger



Design of the strategy: By conditions

- 1) Monotonically increasing ratio for s from $(1,0)$ to $(1,b)$
- 2) Constant ratio for s from $(1,b)$ to $(1,D)$
- Determines a, b and D



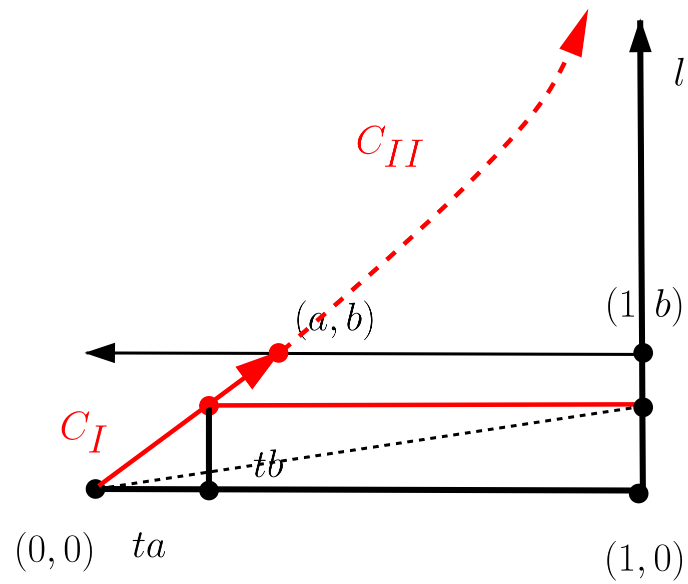
Design of the strategy: Condition 1)

- Start with 1): Ratio for $t \in [0, 1]$:

$$\phi(t) = \frac{t\sqrt{a^2+b^2}+1-ta}{\sqrt{1+t^2b^2}}$$

- Monotonicity: $\phi'(t) \geq 0 \qquad \forall t \in [0, 1]$
- Analysis:
  $\Leftrightarrow \sqrt{a^2 + b^2} - a \geq tb^2 \qquad \forall t \in [0, 1]$
- $\Leftrightarrow b^2 \leq 1 - 2a$
- Choose: $a = \frac{1-b^2}{2}$
- Worst-case ratio:

$$C = \frac{\sqrt{a^2+b^2}+1-a}{\sqrt{1+b^2}} = \sqrt{1 + b^2}$$



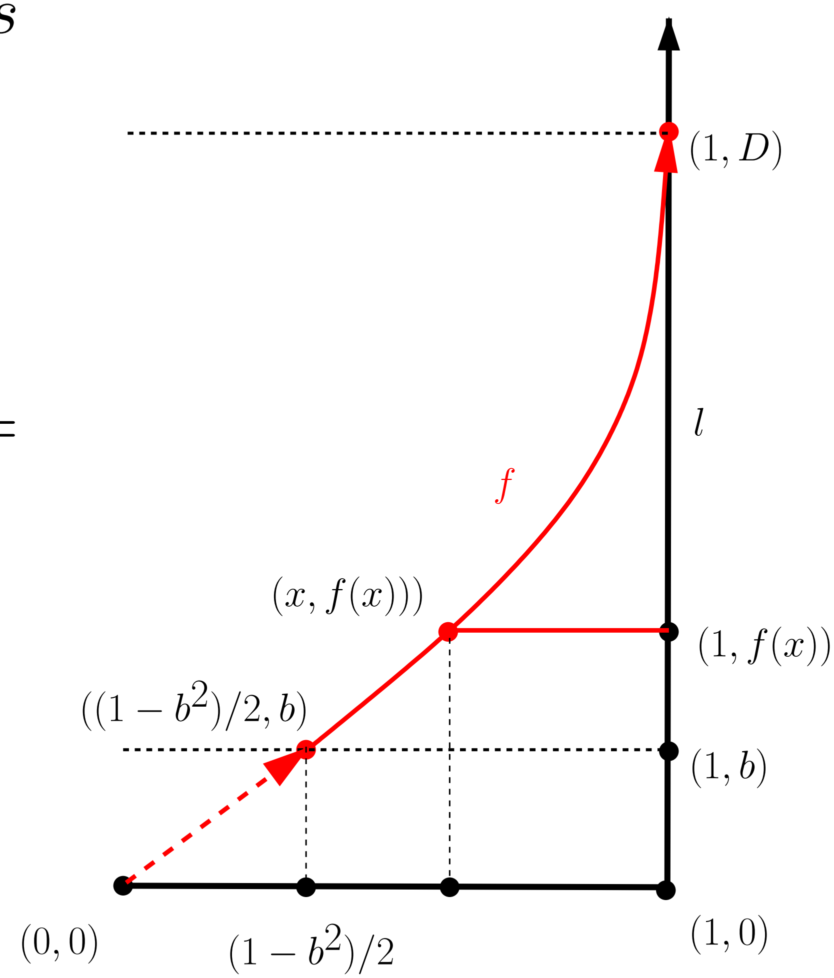
Design of the strategy: Condition 2)

- 2) Constant ratio $C = \sqrt{1+b^2}$ for s from $(1,b)$ to $(1,D)$
- Function $f(x)$ for $x \in [a,1]$
- Constant ratio C:

$$\sqrt{a^2+b^2} + \int_a^x \sqrt{1+f'(t)^2}dt + 1 - x = C \cdot \sqrt{1+f(x)^2}$$

- Transformations ($f'(x) \neq 0!$):

$$\Leftrightarrow f'(x) = 2C \frac{\sqrt{1+f(x)^2} f(x)}{1+(1-C^2)f(x)^2}$$



Solutions for $y = f(x)$

- $f'(x) = 2\sqrt{1+b^2}\frac{\sqrt{1+f(x)^2}f(x)}{1-b^2f(x)^2}$, $((1-b^2)/2, b)$ on the curve
- Solve: $y' = 1 \cdot 2\sqrt{1+b^2}\frac{\sqrt{1+y^2}y}{1-b^2y^2}$ for y with $((1-b^2)/2, b)$
- First order diff. eq. $y' = h(x)g(y)$, separated variables, point (k, l)
- Solution: $\int_l^y \frac{dt}{g(t)} = \int_k^x h(z)dz$

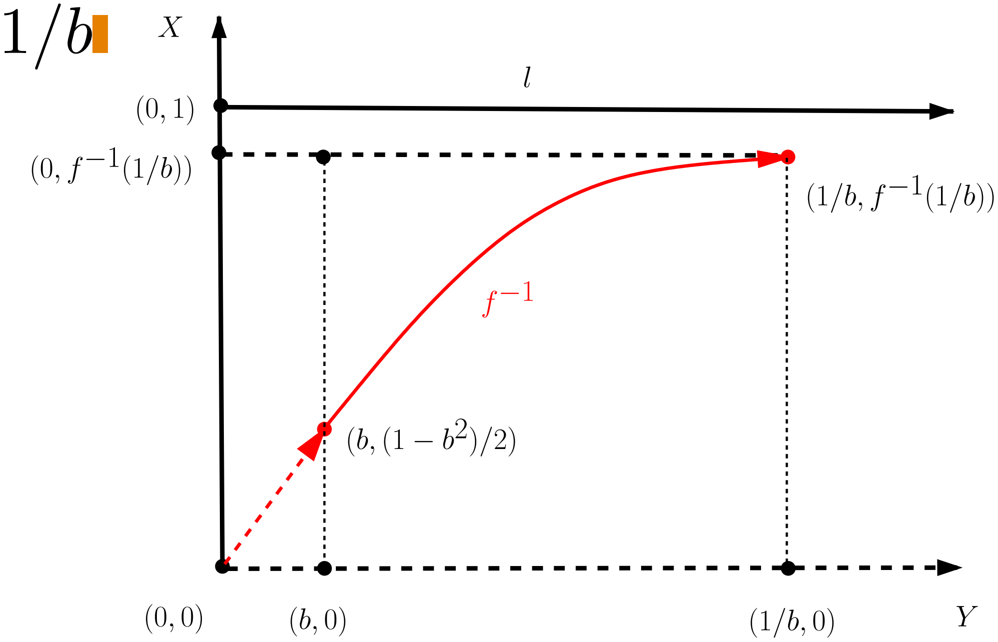
$$\int_b^y \frac{1-b^2t^2}{2\sqrt{1+b^2}\sqrt{1+t^2}t}dt = \int_{(1-b^2)/2}^x 1 \cdot dz = x - (1-b^2)/2$$

$$x = -\frac{b^2\sqrt{1+y^2} + \text{arctanh}\left(1/(\sqrt{1+y^2})\right) - \text{arctanh}\left(1/(\sqrt{1+b^2})\right) - \sqrt{1+b^2}}{2\sqrt{1+b^2}}$$

- Solution for inverse function $x = f^{-1}(y)$, for $y \in [b, 1/b]$

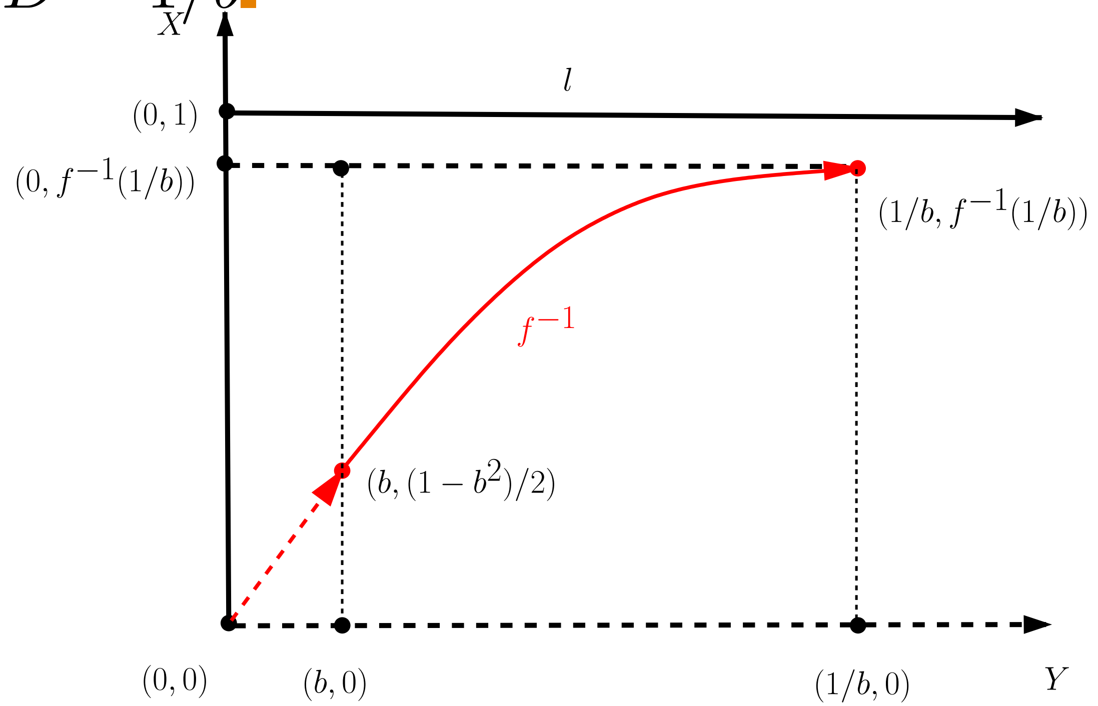
Consider inverse function $x = f^{-1}(y)$

- $x' = \frac{1}{g(y)} = -\frac{(b^2y^2-1)}{2\sqrt{1+y^2}y\sqrt{(1+b^2)}} \geq 0$ for $y \in [b, 1/b]$
- $x'' = -\frac{(b^2y^2+2y^2+1)}{2(1+y^2)^{3/2}\sqrt{1+b^2y^2}} \leq 0$ for $y \geq 0$
- $x = f^{-1}(y)$ concave, $y = f(x)$ convex
- Max. at $y = 1/b$



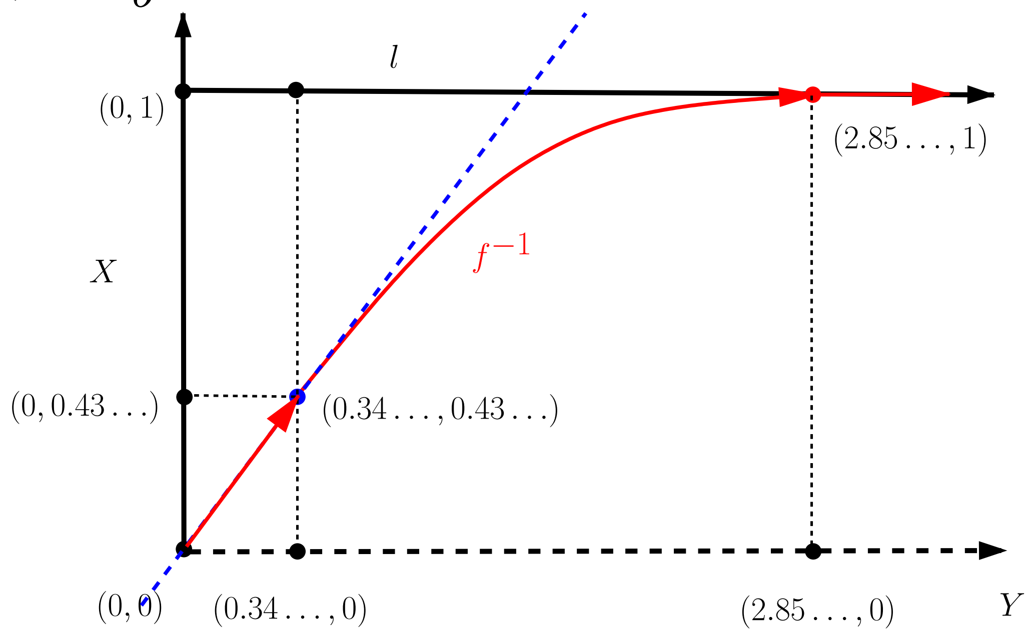
Consider inverse function $x = f^{-1}(y)$

- Maximum at $y = 1/b$
- Find b so that $f^{-1}(1/b) = 1$
- Fixes b and $D = 1/b$



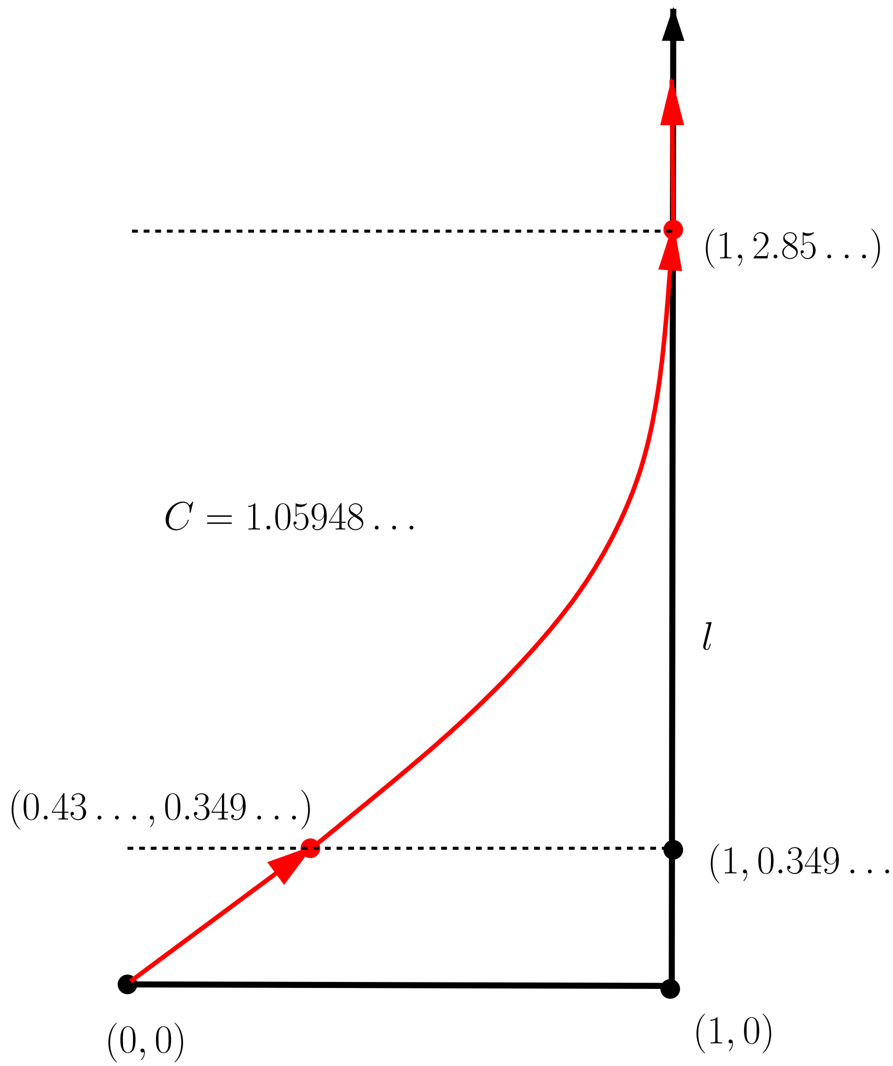
Optimality of f (or f^{-1})

- Solve $f^{-1}(1/b) = 1$: $b = 0.3497\ldots$, $D = 1/b = 2.859\ldots$, $a = 0.43\ldots$, worst-case ratio $C = \sqrt{1+b^2} = 1.05948\ldots$
- f convex from (a,b) to $(1,D)$, line segment convex
- Prolongation of line segment is tangent of f^{-1} at (b,a)
- Insert: $f^{-1\prime}(b) = \frac{a}{b}$!



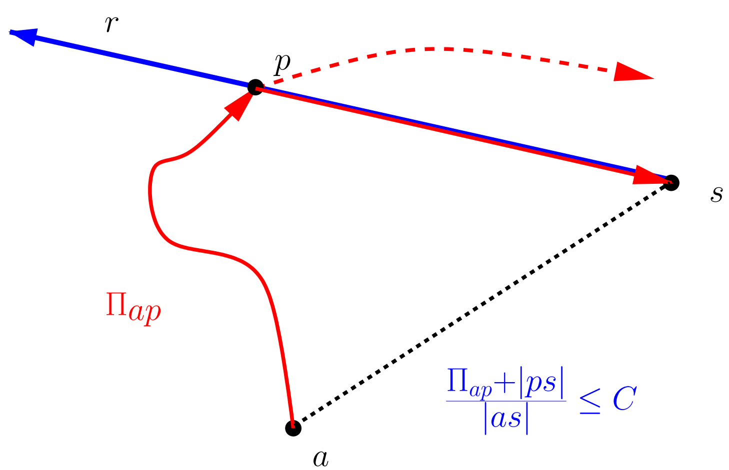
Conclusion

- Optimal strategy with ratio
  $C = 1.05948\ldots$



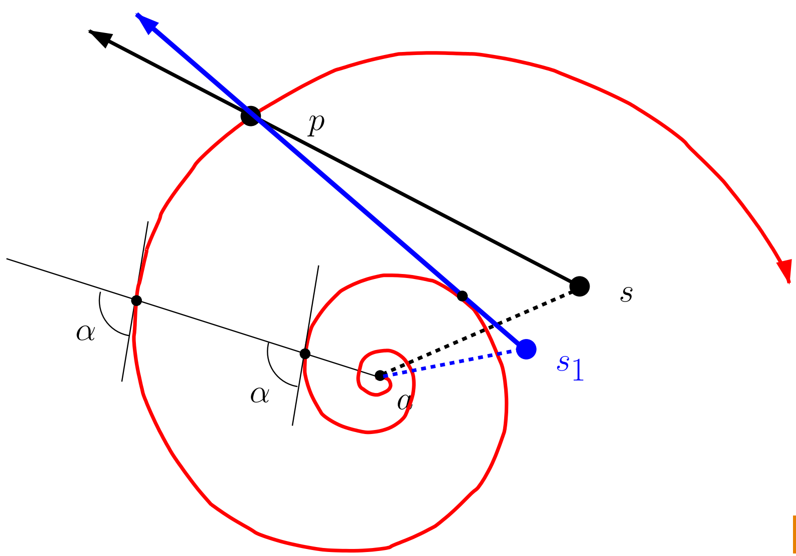
Rays in general

- Rays are somewhere in the plane
- Searchpath Π
- Upper bound: $C = 22.531\ldots$
- Lower bound: $C \geq 2\pi\, e = 17.079\ldots$



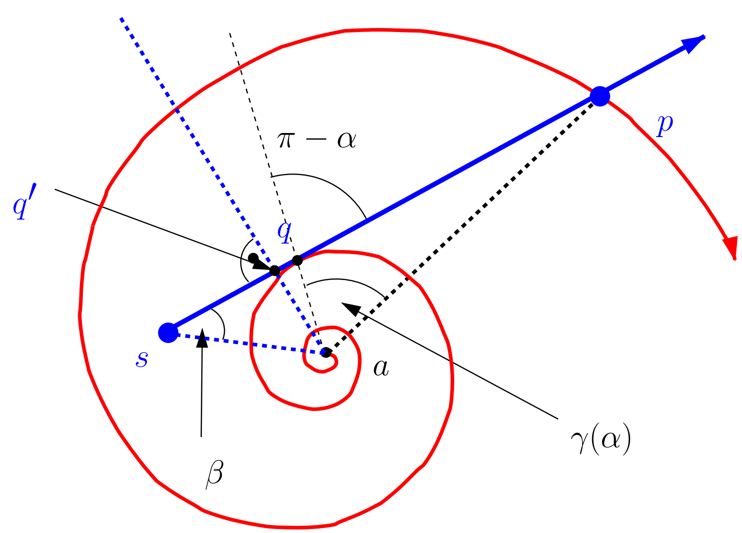
Strategy: Spiral search

- Logarithmic spiral
- Polar coordinates $(\varphi, d \cdot E^{\phi \cot(\alpha)})$, $d > 0, -\infty < \varphi < \infty$
- $\alpha = \pi/2$ Kreis!
- Hits the ray, moves to s
- Worst-case ratio: Ray is a tangent **Lemma**



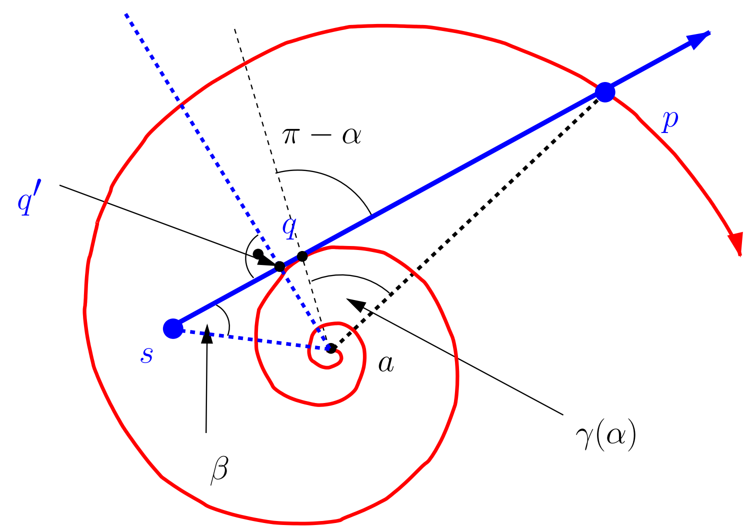
Optimizing the spiral

- Strategy $d \cdot E^{\phi \cot(\alpha)}$, property $|\mathrm{SP}_{\mathrm{a}}^{\mathrm{p}}| = \frac{|\mathrm{ap}|}{|\cos(\alpha)|} = \frac{\mathrm{d}E^{\theta_{\mathrm{p}} \cot \alpha}}{|\cos(\alpha)|}$
- Ratio C identical for all tangents: Ratio $C(\alpha)$
- We optimize for perpendicular points q'
- Adversary can move s a bit to the left (chooses β)
- Law of sine: Ratio $C(\beta, \alpha) = C(\alpha)\sin(\beta) + \cos(\beta)$



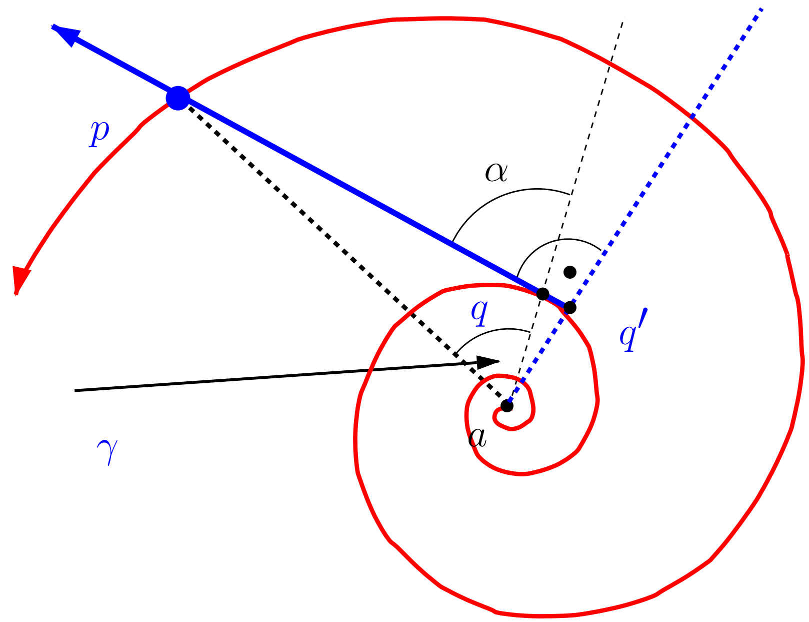
Minimize worst-case ratio

- $dE^{\phi \cot(\alpha))}$: Determine α, $d = 1$
- Assume: Fixed α for q'
- $|as| \sin(\beta) = |aq'|$,
  $|sq'| = |as| \cos(\beta) = \frac{|aq'| \cos(\beta)}{\sin(\beta)}$
- Ratio $C_{q'} = \frac{|\mathrm{SP}_a^p| + |\mathrm{pq}'|}{|aq'|}$
  maximized by
  $\frac{C_{q'} \cdot |aq'| + |aq'| \cos(\beta) / \sin(\beta)}{|aq'| / \sin(\beta)} =$
  $C_{q'} \sin(\beta) + \cos(\beta)$
- Minimize over α for q'
- Finally adverary choose β, fixes s!



Minimize worst-case for q'

- $p = (\phi, E^{\phi \cot(\alpha)})$: Determine $\alpha < \pi/2$
- $|\mathrm{SP}_{\mathrm{a}}^{\mathrm{p}}| = \frac{|\mathrm{ap}|}{\cos(\alpha)} = \frac{\mathrm{E}^{\cot\alpha(2\pi+\gamma+\theta_{\mathrm{q}})}}{\cos(\alpha)}$
- $|pq|\sin(\alpha) = |ap|\sin(\gamma)$ and $|pq| = \frac{E^{\cot\alpha(2\pi+\gamma+\theta_q)}\sin(\gamma)}{\sin\alpha}$
- $|qq'| = |aq|\cos(\alpha)$
- $|pq'| = \frac{E^{\cot\alpha(2\pi+\gamma+\theta_q)}\sin(\gamma)}{\sin\alpha} + E^{\cot\alpha(\theta_q)}\cos(\alpha)$
- $|aq'| = |aq|\sin(\alpha) = E^{\cot\alpha(\theta_q)}\sin(\alpha)$
- γ depends only on α: Now $\gamma(\alpha)$



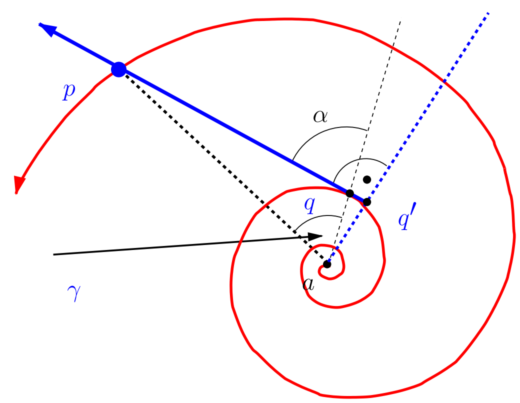
Minimize worst-case for q'

Determine $C_{q'}(\alpha) = \frac{|\Pi_a^p| + |pq'|}{|aq'|}$!

$$\frac{\frac{1}{\cos\alpha}\,\mathrm{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))} + \frac{1}{\sin\alpha}\,\mathrm{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))}\sin\gamma(\alpha) + \mathrm{E}^{\cot\alpha\theta_q}\cos\alpha}{\mathrm{E}^{\cot\alpha\theta_q}\sin\alpha} =$$

$$\frac{\frac{1}{\cos\alpha}\,\mathrm{E}^{\cot\alpha(2\pi+\gamma(\alpha))} + \frac{1}{\sin\alpha}\,\mathrm{E}^{\cot\alpha(2\pi+\gamma(\alpha))}\sin\gamma(\alpha) + \cos\alpha}{\sin\alpha} =$$

$$\left(\frac{1}{\sin\alpha\cdot\cos\alpha} + \frac{\sin\gamma(\alpha)}{\sin^2\alpha}\right)\mathrm{E}^{b(2\pi+\gamma(\alpha))} + \cot\alpha$$



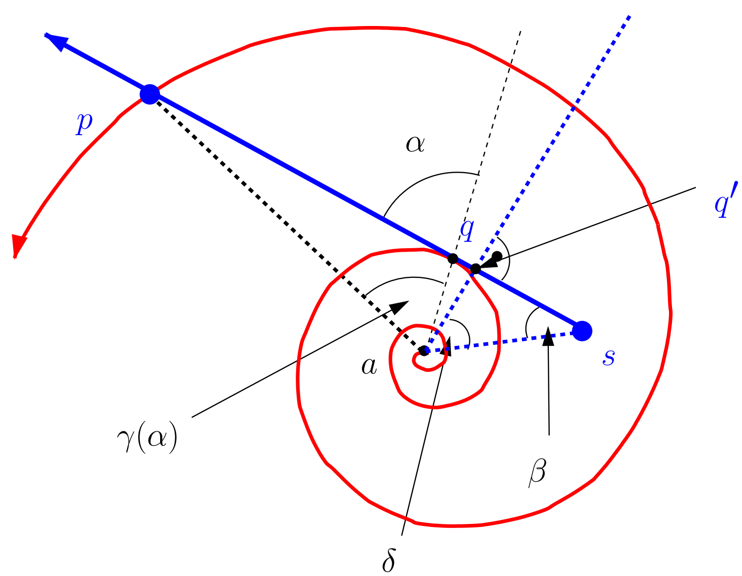
Minimize worst-case for $s = q'$

Determine $\gamma(\alpha)$, Exercise!

Solve Equation: $\frac{\sin\alpha}{\sin(\alpha-\gamma(\alpha))} = E^{\cot\alpha(2\pi+\gamma(\alpha))}$

Then optimize: $f(\alpha) := \left(\frac{1}{\sin\alpha\cdot\cos\alpha} + \frac{\sin\gamma(\alpha)}{\sin^2\alpha}\right) \mathrm{E}^{b(2\pi+\gamma(\alpha))} + \cot\alpha$

Then minimize: $g(\beta) := f(\alpha_{\min})\sin\beta + \cos\beta$!



Optimizing the spiral: Theorem 2.42

- Ratio: $C(\alpha)$ for $s = q'$ minimal for $\alpha = 1.4575\ldots$
- $C(\alpha) = 22.4908\ldots$
- Adversary choose β for max. $D(\beta,\alpha) = C(\alpha)\sin(\beta) + \cos(\beta)$
- For $\alpha = 1.4575\ldots$

  choose $\delta = 0.044433\ldots$

  $D(\beta, C(\alpha)) = 22.51306056\ldots$