Problem Set 6

Problem 1

Give examples for the following situations or reason that they cannot occur.

1. Draw paths $P(i)$ and $P(j)$ and give a value $t$ such that $\text{offset}(t, j) = 2|P(i)|$.

2. Draw a path $P(i)$ and possibly other intersecting paths and give a value $t$ such that $\text{offset}(t, i) = 2|P(i)|$.

3. Draw a path $P(i)$ with $|P(i)| = 4$ and a path $P(j)$ and give a value $L$ such that $\text{lucky}(L+1, L) = \emptyset$, $\text{lucky}(t, L) = \{j\}$ for all times $t \in \{L+2, \ldots, |P(i)|+L\}$.

4. Draw a path $P(i)$ and possibly other intersecting paths and give a value $L$ such that $\text{lucky}(t, L) \neq \emptyset$ and $\text{lucky}(t', L) = \emptyset$ for all $t, t' \in \{L+1, \ldots, |P(i)|+L\}$ with $t \neq t'$.

5. Draw paths $P(i)$ and $P(j)$ and give values $t$ and $L$ such that $\text{lucky}(t, L) = \emptyset$ and $\text{lucky}(t+1, L) = \emptyset$ and $\text{lucky}(t+2, L) = \{j\}$.

Problem 2

Assume that we throw $m = \lfloor 5n \log_2 n \rfloor$ balls into $n$ baskets. For each ball, the basket is chosen uniformly at random. Let $X_i$ be the number of balls in basket $i$ for $i \in \{1, \ldots, n\}$, and set $X = \max_{i=1,\ldots,n} X_i$. Show that

$$\text{Pr}(X \geq 30 \log_2 n) \leq \frac{1}{n^c}$$

holds for a suitable constant $c > 0$.

Problem 3

Show that the recurrence

$$h_0 = 1 + h_1, \quad h_n = 0, \quad \forall j \in \{0, \ldots, n-1\} : h_j \leq 1 + \frac{2}{3}h_{j-1} + \frac{1}{3}h_{j+1}$$

implies that $h_j \leq 2^{n+j^2-2j^2-3(n-j)}$ for all $j \in \{0, \ldots, n\}$. Hint: First show by induction that for all $j \in \{0, \ldots, n-1\}$, $h_j \leq h_{j+1} + 2^{j^2-3} - 3$ holds.

(This task completes the proof of Lemma 4.3).