Online Motion Planning MA-INF 1314
Searching in streets!

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Rep.: Street

Def. Polygonal boundary chains $P_L$ and $P_R$ of $P$ between $s$ and $t$ weakly visible.

Task: Start at $s$, find $t$!
Rep.: Lower Bound!

Theorem: No strategy attains a ratio better than $\sqrt{2}$ versus the length of the shortest path.

Beweis:

Detour of factor at least $\sqrt{2}$

![Diagram](image-url)
Rep.: Reasonable strategies!

- Rightmost left reflex vertex, leftmost right reflex vertex!
- Move into the wedge of $c$, $v_l$ and $v_r$.
- One side-candidate vanishes, move directly to the other.
- Extreme vertices change over time.

\[ v_r = v_1 \quad v_l = v_1 \]
Rep.: Funnel polygons!

- It is sufficient to analyse special streets.

- **Def.** Polygon, conve vertex $s$, two opening convex polygonal chains $P_L$ and $P_R$ starting in $s$ ending at $t_\ell$ and $t_r$, respectively. Segment $\overline{t_\ell t_r}$ closes the funnel (polygon).
Lemma: LB for funnel of opening angle $\phi$: $K_\phi := \sqrt{1 + \sin \phi}$.

- Strongly increasing: $0 \leq \phi \leq \pi/2$, Interval $[1, \sqrt{2}]$
- Strongly decreasing: $\pi/2 \leq \phi \leq \pi$, Interval $[\sqrt{2}, 1]$
- Subdivide: Strategy up to $\phi_0 = \pi/2$, Strategy from $\phi_0 = \pi/2$
Rep.: Opt. strat. for angles $\pi \geq \phi_0 \geq \pi/2$

- Backward analysis: For $\varphi_n := \pi$ optimal strategy.
- $K_\pi = 1$ and $K_\pi$-competitive opt. strategy with path $l_n$ or $r_n$.
- Assumption: Opt. strategy for some $\phi_2$ with factor $K_{\phi_2}$ ex.
- How to prolong for $\phi_1$ with factor $K_{\phi_1}$ where $\pi/2 \leq \phi_1 < \phi_2$?
- We have $K_{\phi_1} > K_{\phi_2}$.
Opt. strat. opening angle $\pi \geq \varphi_0 \geq \pi/2!$

- Situation: Opt. strategy for $\phi_2$ with ratio $K_{\phi_2}$
- How to get opt. strategy for $K_{\phi_1}$?
- Conditions for the path $w$? Design!
- Goal behing $v_l$, path: $|w| + K_{\phi_2} \cdot \ell_2$, optimal: $l_1$
- Goal behind $v_r$, path: $|w| + K_{\phi_2} \cdot r_2$, optimal: $r_1$
- Means: $\frac{|w| + K_{\phi_2} \cdot \ell_2}{l_1} \leq K_{\phi_1}$ and $\frac{|w| + K_{\phi_2} \cdot r_2}{r_1} \leq K_{\phi_1}$
Opt. strat. opening angle $\pi \geq \varphi_0 \geq \pi/2$!

- Guarantee: $|w| + K\phi_2\cdot\ell_2 \leq K\phi_1$ and $|w| + K\phi_2\cdot r_2 \leq K\phi_1$
- Combine, single condition for $w$
- $|w| \leq \min\{ K\phi_1\ell_1 - K\phi_2\ell_2, K\phi_1r_1 - K\phi_2r_2 \}$
- Change of a vertex at $p_2$? Remains guilty!
Opt. strat. opening angle $\pi \geq \varphi_0 \geq \pi/2!$

- Change left hand: Condition
  
  \[|w| \leq \min\{ K_{\phi_1} \ell_1 - K_{\phi_2} \ell_2, K_{\phi_1} r_1 - K_{\phi_2} r_2 \}\]

- There is opt. strategy for $\phi_2$

- Show: \[|w| + K_{\phi_2} (\ell_2 + \ell'_2) \leq K_{\phi_1}\]
Opt. strat. opening angle $\pi \geq \varphi_0 \geq \pi/2$!

\[
|w| \leq K_{\phi_1} l_1 - K_{\phi_2} l_2 \\
= K_{\phi_1} l_1 - K_{\phi_2} l_2 + K_{\phi_2} l'_2 - K_{\phi_2} l'_2 \\
\leq K_{\phi_1} (l_1 + l'_2) - K_{\phi_2} (l_2 + l'_2)
\]
Opt. strat. opening angle $\pi \geq \varphi_0 \geq \pi/2!$

**Lemma** Let $S$ be a strategy for funnels with opening angles $\phi_2 \geq \pi/2$ and competitive ratio $K_{\phi_2}$. We can extend this strategy to a strategy with ratio $K_{\phi_1}$ for funnels with opening angles $\phi_1$ where $\phi_2 > \phi_1 \geq \pi/2$, if we guarantee

$$|w| \leq \min\{ K_{\phi_1}l_1 - K_{\phi_2}l_2, K_{\phi_1}r_1 - K_{\phi_2}r_2 \}$$

for the path $w$ from $p_1$ (opening angle $\phi_1$) to $p_2$ (opening angle $\phi_2$).
Opt. strat. opening angle $\pi \geq \varphi_0 \geq \pi/2!$

- If $|w| \leq \min\{K\phi_1\ell_1 - K\phi_2\ell_2, K\phi_1r_1 - K\phi_2r_2\}$ holds, then
- $|W| \leq \min\{K\phi_0 \cdot |P_L| - K\pi\ell_{\text{End}}, K\phi_0 \cdot |P_R| - K\pi r_{\text{End}}\}$.
Opt. strat. opening angle $\pi \geq \varphi_0 \geq \pi/2$!

- $|w| \leq \min\{ K_{\phi_1} \ell_1 - K_{\phi_2} \ell_2 , K_{\phi_1} r_1 - K_{\phi_2} r_2 \}$
- How to fulfil this?
- Equality for both sides: $K_{\phi_2} (\ell_2 - r_2) = K_{\phi_1} (\ell_1 - r_1)$
- Good choice for both sides!
- Defines a curve!
- We start with $A = K_{\phi_0} (\ell_0 - r_0)$
- Parametrisation!
\[ A = K_{\phi_0} (\ell_0 - r_0) \]

- **Hyperbola:** \( \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1, \ l - r = 2a, \ 2c, \ a^2 + b^2 = c^2 \)

- **Circle:** \( X^2 + (Y - x)^2 = z^2, \ r = z, \ (0, x) \)
Intersection with circle and hyperbola

- Hyperbola: \( \frac{X^2}{\left(\frac{A}{2K\phi}\right)^2} - \frac{Y^2}{\left(\frac{1}{2}\right)^2} - \left(\frac{A}{2K\phi}\right)^2 = 1 \)

- Circle: \( X^2 + \left(Y + \frac{\cot \phi}{2}\right)^2 = \frac{1}{4\sin^2 \phi} \)
Opt. strat. for opening angle $\pi \geq \varphi_0 \geq \pi/2$!

Intersection: Verification by insertion!

$$X(\phi) = \frac{A}{2} \cdot \cot \frac{\phi}{2} \cdot \sqrt{\left(1 + \tan \frac{\phi}{2}\right)^2} - A^2$$

$$Y(\phi) = \frac{1}{2} \cdot \cot \frac{\phi}{2} \cdot \left(\frac{A^2}{1 + \sin \phi} - 1\right)$$

where $A = K\varphi_0(\ell_0 - r_0)$
Opt. strat. for opening angle \( \pi \geq \phi_0 \geq \pi/2! \)

\[
X(\phi) = \frac{A}{2} \cdot \cot \frac{\phi}{2} \cdot \sqrt{\left(1 + \tan \frac{\phi}{2}\right)^2 - A^2}
\]

\[
Y(\phi) = \frac{1}{2} \cdot \cot \frac{\phi}{2} \cdot \left(\frac{A^2}{1 + \sin \phi} - 1\right)
\]

Change of the boundary points. \( A \) also changes, new piece of curve!
Opt. strat. for opening angle $\pi \geq \varphi_0 \geq \pi/2$!

**Theorem:** The goal of a funnel with opening angle $\phi_0 > \pi/2$ can be found with ratio $K_{\phi_0}$.

Proof: Show that the curves fulfil:

$$|w| \leq \min\{ K_{\phi_1} \ell_1 - K_{\phi_2} \ell_2, K_{\phi_1} r_1 - K_{\phi_2} r_2 \}$$

For any small piece $w$ of the curve. Analytically, lengthy proof! Experimentally!
Opt. strat. opening angle $0 \leq \varphi_0 \leq \pi/2$!

- The same approach
- But independent from the angle
- Dominated by factor $K_{\pi/2} = \sqrt{2}$
- Require: $w \leq \min\{ \sqrt{2}(\ell_1 - \ell_2), \sqrt{2}(r_1 - r_2) \}$
- Equality: $\ell_1 - \ell_2 = r_1 - r_2$
- Current angular bisector: Hyperbola!

![Diagram of angular bisector and distances](image)
Opt. strat. opening angle $0 \leq \varphi_0 \leq \pi$!

Combine strategy 1 and strategy 2

Theorem: In an unknown street-polygon beginning from the source $s$ we can find the target $t$ with an optimal online strategy with competitive ratio $\sqrt{2}$.
Optimal strategy “Worst-Case-Aware”

As long as target $t$ is not visible:
- Compute current $v_\ell$ and $v_r$.
- If only one exists: Move directly toward the other.
- Otherwise. Repeat:
  - New reflex vertex $v'_\ell$ or $v'_r$ is detected:
    - Use $v'_\ell$ or $v'_r$ instead of $v_\ell$ or $v_r$.
  - Let $\phi$ be the angle between $v_\ell$, the current position and $v_r$.
    - If $\phi \leq \frac{\pi}{2}$: Follow the current angular bisector!
    - If $\phi > \frac{\pi}{2}$: Follow the curve $(X(\phi), Y(\phi))$.
  - Until either $v_\ell$ or $v_r$ is explored.
  - Move toward the non-explored vertex.
- Move toward the goal.