Online Motion Planning MA-INF 1314

General rays!

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Rep.: Window Shopper Strategy: Three parts

- A line segment from \((0, 0)\) to \((a, b)\) with increasing ratio for \(s\) between \((1, 0)\) and \((1, b)\)
- A curve \(f\) from \((a, b)\) to some point \((1, D)\) on \(l\) which has the same ratio for \(s\) between \((1, b)\) and \((1, D)\)
- A ray along the \textit{window} starting at \((1, D)\) with decreasing ratio for \(s\) beyond \((1, D)\) to infinity
- Worst-case ratio is attained for all \(s\) between \((1, b)\) and \((1, D)\)
- Optimality by construction, CONVEX!
Rep.: Design of the strategy

- Condition I): \[ a = \frac{1-b^2}{2}, \quad \sqrt{1+b^2} \]
- Condition II):
  \[ f'(x) = 2\sqrt{1+b^2} \frac{\sqrt{1+f(x)^2f(x)}}{1-b^2 f(x)^2}, \]
  Point \(((1-b^2)/2, b)\)
- Solve: \[ y' = 1 \cdot 2\sqrt{1+b^2} \frac{\sqrt{1+y^2y}}{1-b^2 y^2} \text{ with} \]
  \[ ((1-b^2)/2, b) \]
- First order diff. eq. \[ y' = h(x)g(y) \]
- Solution: \[ \int_l^y \frac{dt}{g(t)} = \int_k^x h(z)dz: \]
  Here \[ x = f^{-1}(y) \]
Rep.: Consider inverse function \( x = f^{-1}(y) \)

- \( x = -\frac{b^2 \sqrt{1+y^2} + \text{arctanh}\left(\frac{1}{\sqrt{1+y^2}}\right) - \text{arctanh}\left(\frac{1}{\sqrt{1+b^2}}\right)}{2 \sqrt{1+b^2}} \)

- \( x' = \frac{1}{g(y)} = -\frac{(b^2 y^2 - 1)}{2 \sqrt{1+y^2} y \sqrt{(1+b^2)}} \geq 0 \) for \( y \in [b, 1/b] \)

- \( x'' = -\frac{(b^2 y^2 + 2 y^2 + 1)}{2 (1+y^2)^{3/2} \sqrt{1+b^2} y^2} \leq 0 \) for \( y \geq 0 \)

- \( x = f^{-1}(y) \) concave, \( y = f(x) \) convex, Max. at \( y = 1/b \)
Rep.: Theorem Optimality of $f$

- Solve $f^{-1}(1/b) = 1$: $b = 0.3497\ldots$, $D = 1/b = 2.859\ldots$, $a = 0.43\ldots$, worst-case ratio $C = \sqrt{1 + b^2} = 1.05948\ldots$
- $f$ convex from $(a, b)$ to $(1, D)$, line segment convex
- Prolongation of line segment is tangent of $f^{-1}$ at $(b, a)$
- Insert: $f^{-1}'(b) = \frac{a}{b}$! Convex!
Rays in general

- Rays are somewhere in the plane
- Searchpath $\Pi$
- Upper bound: $C = 22.531\ldots$
- Lower bound: $C \geq 2\pi e = 17.079\ldots$
Strategy: Spiral search

• Logarithmic spiral
• Polar coordinates \((\varphi, d \cdot E^\phi \cot(\alpha))\), \(d > 0, -\infty < \varphi < \infty\)
• \(\alpha = \pi/2\) Kreis!
• Hits the ray, moves to \(s\)
• Worst-case ratio: Ray is a tangent Lemma 2.38
Optimizing the spiral

- Strategy $d \cdot E^\phi \cot(\alpha)$, property $|\text{SP}_a^p| = \frac{|ap|}{\cos(\alpha)} = \frac{dE^\theta_p \cot \alpha}{\cos(\alpha)}$

- Ratio $C$ identical for all tangents: Ratio $C(\alpha)$

- We optimize for perpendicular points $q'$

- Adversary can move $s$ a bit to the left (chooses $\beta$)

- Law of sine: Ratio $C(\beta, \alpha) = C(\alpha) \sin(\beta) + \cos(\beta)$
Minimize worst-case ratio

- $dE^\Phi \cot(\alpha)$: Determine $\alpha$, $d = 1$
- Assume: Fixed $\alpha$ for $q'$
- $|as\sin(\beta)| = |a q'|$
- $|s q'| = |a s| \cos(\beta) = \frac{|a q'| \cos(\beta)}{\sin(\beta)}$
- Ratio $C_{q'} = \frac{|SP_a| + |pq'|}{|a q'|}$ maximized by
  $\frac{|a q'|/|a q'|\sin(\beta)}{\cos(\beta)} = C_{q'} \sin(\beta) + \cos(\beta)$
- Minimize over $\alpha$ for $q'$
- Finally adversary choose $\beta$, fixes $s$!
Minimize worst-case for $q'$

- $p = (\phi, E^{\phi \cot(\alpha)})$: Determine $\alpha < \pi/2$

- $|SP^p_\alpha| = \frac{|ap|}{\cos(\alpha)} = \frac{E^{\cot(\alpha)(2\pi+\gamma+\theta q)}}{\cos(\alpha)}$

- $|pq| \sin(\alpha) = |ap| \sin(\gamma)$ and $|pq| = \frac{E^{\cot(\alpha)(2\pi+\gamma+\theta q)}}{\sin(\alpha)} \sin(\gamma)$

- $|qq'| = |aq| \cos(\alpha)$

- $|pq'| = \frac{E^{\cot(\alpha)(2\pi+\gamma+\theta q)}}{\sin(\alpha)} \sin(\gamma) + E^{\cot(\alpha)(\theta q)} \cos(\alpha)$

- $|aq'| = |aq| \sin(\alpha) = E^{\cot(\alpha)(\theta q)} \sin(\alpha)$

- $\gamma$ depends only on $\alpha$: Now $\gamma(\alpha)$
Minimize worst-case for $q'$

Determine $C_{q'}(\alpha) = \frac{|\Pi_{p}^q| + |pq'|}{|aq'|}$

\[
\frac{1}{\cos \alpha} E^{\cot \alpha (\theta_q + 2\pi + \gamma(\alpha))} + \frac{1}{\sin \alpha} E^{\cot \alpha (2\pi + \gamma(\alpha))} \sin \gamma(\alpha) + E^{\cot \alpha \theta_q} \cos \alpha = E^{\cos \alpha \theta_q} \sin \alpha
\]

\[
\frac{1}{\cos \alpha} E^{\cot \alpha (2\pi + \gamma(\alpha))} + \frac{1}{\sin \alpha} E^{\cot \alpha (2\pi + \gamma(\alpha))} \sin \gamma(\alpha) + \cos \alpha = \sin \alpha
\]

\[
\left( \frac{1}{\sin \alpha \cdot \cos \alpha} + \frac{\sin \gamma(\alpha)}{\sin^2 \alpha} \right) E^{b(2\pi + \gamma(\alpha))} + \cot \alpha
\]
Minimize worst-case for $s = q'$

Determine $\gamma(\alpha)$, Exercise!

Solve Equation: $\frac{\sin \alpha}{\sin(\alpha - \gamma(\alpha))} = E^\cot \alpha(2\pi + \gamma(\alpha))$

Then optimize: $f(\alpha) := \left(\frac{1}{\sin \alpha \cdot \cos \alpha} + \frac{\sin \gamma(\alpha)}{\sin^2 \alpha}\right) E^b(2\pi + \gamma(\alpha)) + \cot \alpha$

Then minimize: $g(\beta) := f(\alpha_{\min}) \sin \beta + \cos \beta$
Optimizing the spiral: Theorem 2.42

- Ratio: $C(\alpha)$ for $s = q'$ minimal for $\alpha = 1.4575\ldots$
- $C(\alpha) = 22.4908\ldots$
- Adversary choose $\beta$ for max.
  $D(\beta, \alpha) = C(\alpha) \sin(\beta) + \cos(\beta)$
- For $\alpha = 1.4575\ldots$ choose $\delta = 0.044433\ldots$

$$D(\beta, C(\alpha)) = 22.51306056\ldots$$
Lower bound: Special example

- General problem of lower bounds
- Special example: Searching for a special ray in the plane
- Cross $R$ and detect $s$
- Special case: No move to $s$
- Alpern/Gal (Spiral search): Ratio $C = 17.289\ldots$
- Best ratio among monotone and periodic strategies
- ”Complicated task”: There is an optimal periodic/monotone strategy
Lower bound construction

- $n$ known rays emanating from $a$
- Angle $\alpha = \frac{2\pi}{n}$
- Find $s$ on one of the rays
- Also non-periodic and non-monotone strategies
- Strategy $S$: Visits the rays in some order
- Hit $x_k$, leave $\beta_k x_k$ ($\beta_k \geq 1$)
Lower bound construction

- Find $s$ on a ray visited up to $\beta_k x_k$ at the last time, now at $x_{J_k}$
- Note: Any order is possible
- Worst-case, $s$ close to $\beta_k x_k$
- Ratio: $C(S)$

$$\sum_{i=1}^{J_k-1} \frac{\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos \gamma_{i,i+1} + x_{i+1}^2} + (\beta_i x_i - x_i)^2}{\beta_k x_k}$$

- Monotone/Periodic, Funktional??
Lower bound construction

- Ratio: \( C(S) \)

\[
\sum_{i=1}^{J_k-1} \frac{\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos \gamma_{i,i+1} + x_{i+1}^2} + (\beta_i x_i - x_{i+1}) \beta_{j_k} x_{j_k}}{\beta_k x_k}
\]

- Shortest distance to next ray:

\( \beta_i x_i \sin \frac{2\pi}{n} \)

- Lower bound for

\[
\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos \gamma_{i,i+1} + x_{i+1}^2}
\]

- Lower bound: \( C(S) \geq \)

\[
\sin \frac{2\pi}{n} \sum_{i=1}^{J_k-1} \frac{\beta_i x_i}{\beta_k x_k}
\]
Lower bound construction

- Lower bound \( \sum_{i=1}^{J_k-1} f_i \)
- Equals functional of standard m-ray search
- Optimal strategy: monotone/periodic (Alpern/Gal)
- \( f_i = \left( \frac{n}{n-1} \right)^i \)
- Ratio: \( (n - 1) \left( \frac{n}{n-1} \right)^n \)
- \( C(S') \geq \sin \frac{2\pi}{n} (n - 1) \left( \frac{n}{n-1} \right)^n \)
Lower bound construction

\[ C(S) \geq \sin \frac{2\pi}{n} (n - 1) \left( \frac{n}{n-1} \right)^n \]

\[
\lim_{n \to \infty} (n - 1) \left( \frac{n}{n - 1} \right)^n \sin \frac{2\pi}{n} = 2\pi e = 17.079 \ldots
\]

\* Lower bound: **Theorem**
Summary

• The Window-Shopper-Problem

Optimal strategy $C = 1.059\ldots$: **Theorem**

• Interesting design technique

• Rays in general

• Lower $C \geq 2\pi e = 17.079\ldots$ (**Theorem**) and upper bound $C = 22.51\ldots$ (**Theorem**)

• Lower bound construction

• Also a lower bound for special case with $C = 17.289\ldots$