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side) hold and the left hand side of inequality (3.1) is even smaller now. The remaining inequlities (3.3) are not concerned from this exchange.

The remaining task is to handle the case $J_{j+1} < J_j$. Here we have the problem of maintaining inequality (3.1). To overcome this problem we exchange the role of the rays of f_j und f_{j+1} directly after the index $j+1$ completely. After index $j+1$ any original visit of the ray of f_{j+1} is no applied to the ray of f_j and vice versa. Of course the exchange $f'_j = f_{j+1}$ and $f'_{j+1} = f_j$ is maintained. Now we do not have a problem with inequality (3.1) any more since the ray is visited early enough now. Inequality (3.2) is also maintained because we have the same next visits as before. Inequalities (3.3) do not change, they are not influenced by the exchange. In principle for $J_{j+1} < J_j$ and $f_j > f_{j+1}$ we exchange two complete rays beginning with index j.

For example if $f_1 > f_2$ holds and $J_1 = 7$ for ray K and $J_2 = 5$ for ray L, then after the exchange we visit K by $f'_1 := f_2$ then L by $f'_2 := f_1$, later K by $f'_5 := f_5$ and L by $f'_7 := f_7$ and so on. Figure 3.4 shows an example.

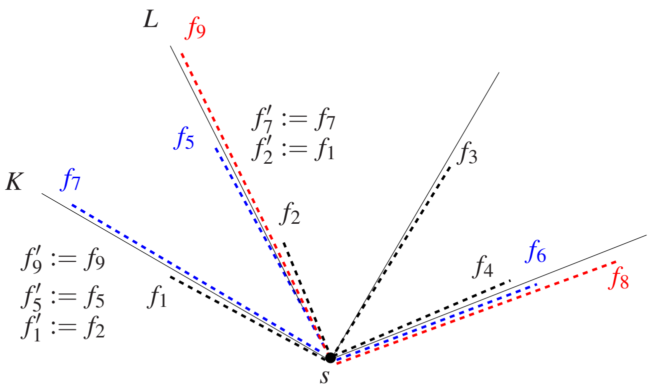

Figure 3.4: A non-periodic and non-monotone strategy. First, we exchange the values f_1 and f_2 only. But since $J_1 = 7 > J_2 = 5$ holds we fully exchange the role for the corresponding rays K and L.

Altogether, we obtain a C-competitive strategy $(f'_1, f'_2, \ldots)$ with $f'_j \leq f'_{j+1}$ for all j by applying the above exchange successively.

Finally, we construct a periodic strategy by the same idea. Consider a monotone strategy with a first index j such that $J_{j+1} < J_j$. We exchange the role of the corresponding rays after step $j+1$, which means that f_j and f_{j+1} remain on their place. Now $J'_{j+1} > J'_j$ holds. The ray with smaller f_j is visited earlier which maintains the ratio, the ray with next at visit J_j s visited later now but the original strategy maintains the ratio for the corresponding sum with f_j and we have f_{j+1} on the right hand side now. All other inequalities are not concerned.

Now after this change it might happen that some the monotonicity after step $j+1$ is no longer given. Then we apply the first rearrangement again and so on.

Altogether we obtain a monotone strategy with $J_{j+1} > J_j$ for all j and the same ratio C. Trivially, if $J_{j+1} > J_j$ holds for all j, this can only mean that $J_j = j+m$ holds for all j. □

3.1.2 Alternative approach: Equality

By Theorem 3.2 we obtained a very general approach for solving motion planning problems optimally. Somehow we can call this the *Optimality of exponential functions*. Here we would like to present an alternative. The corresponding paradigm can be denoted as *Optimality by equality*. Consider the 2-ray search problem. The optimal strategy $x_i = 2^{i-1}$ attains the competitive ratio asymptotically, but never reaches the ratio exactly. Even after the first round we attain $2x_1 + 1 \leq C \cdot 1$ and for $x_1 = 1$ and $C = 9$ there is some room for the first step. Interestingly the strategy $x_i = (i+1)2^i$ fulfils the inequality $\sum_{i=1}^{k+1} x_i \leq \frac{C-1}{2} x_k$ by equality for all k. This can be easily shown by induction.

Exercise 20 *Show that the strategy $x_i = (i+1)2^i$ attains a competitive ratio of $C = 9$ and fulfils $\sum_{i=1}^{k+1} x_i = \frac{C-1}{2} x_k$ for all k.*

We will now show that this is not given by chance. Assume that there is an optimal C-competitive strategy for the 2-ray search problem. This means that there is a sequence $(x_1, x_2, \ldots)$ such that $\frac{\sum_{i=1}^{k+1} x_i}{x_k} \leq \frac{(C-1)}{2}$ holds for all k. In this case there is also always a strategy $(x'_1, x'_2, x'_3 \ldots)$ with $\frac{\sum_{i=1}^{k+1} x'_i}{x'_k} = \frac{(C-1)}{2}$ for all k. The proof for this statement works as follows: The functional $\frac{\sum_{i=1}^{k+1} x_i}{x_k}$ is strictly monotonically decreasing for x_k. If we increase x_k the functional decreases. On the other hand the $\frac{\sum_{i=1}^{k+1} x_i}{x_k}$ is strictly increasing for all x_j with $j \neq k$. This means that increasing x_j for $j \neq k$ will increase the functional.

We can assume that $x_i \geq 0$ holds. Now assume that there is a first index k such that inequality holds, i.e.: $\frac{\sum_{i=1}^{k+1} x_i}{x_k} < \frac{(C-1)}{2}$. If this holds already for the first index $k = 0$ with $x_0 := 1$ we simply decrease x_0 such that $x_1 + 1 = \frac{(C-1)}{2} x_0$ holds. Now consider $k > 0$ as the smallest index with $\frac{\sum_{i=1}^{k+1} x_i}{x_k} < \frac{(C-1)}{2}$. We decrease x_k in such a ways that $\frac{\sum_{i=1}^{k+1} x_i}{x_k} = \frac{(C-1)}{2}$ is given, which is possible in this case. All other inequalities remain valid. Obviously, we will attain $\frac{\sum_{i=1}^{k} x_i}{x_{k-1}} < \frac{(C-1)}{2}$ for the index $k-1$ and we will proceed for $k-1$ by the same argument until finally we reach $k = 0$ again. Note that by $x_1 + 1 = \frac{(C-1)}{2} x_0$, x_0 cannot decrease to 0. Thus for any such first index k we attain a monotonically decreasing sequence $(x_0, x_1, \ldots, x_k)$ that is bounded. This means that the above procedure will always converge to a sequence $(x_0, x_1, \ldots, x_k)$ such that $\frac{\sum_{i=1}^{j+1} x_i}{x_j} = \frac{(C-1)}{2}$ for $j = 0$ to k. This holds for any k. This means that for infinitely many steps there always exists a strategy that achieves equality in any step. Finally, we make use of a scalar A such that $A \cdot x_0 = 1$ holds and $(1, A \cdot x_1, \ldots, A \cdot x_k)$ is the desired sub-strategy.

The above arguments show that such a strategy exists for any k but the procedure is not constructive. Let us now assume that $\frac{\sum_{i=1}^{k+1} x_i}{x_k} = \frac{(C-1)}{2}$ holds for all k, we conclude $x_{k+1} = \sum_{i=1}^{k+1} x_i - \sum_{i=1}^{k} x_i = \frac{(C-1)}{2}(x_k - x_{k-1})$ and we are searching for the solution of the recurrence $\frac{(C-1)}{2}(x_k - x_{k-1}) = x_{k+1}$. We set $x_1 = \frac{(C-1)}{2} = \frac{(C-1)}{2}(x_0 - x_{-1})$ with starting values $x_0 = 1$ and $x_{-1} = 0$. The task is to find the smallest value for C such that the above recurrence attains a reasonable solution for the 2-ray search problem. This means that we will have a look at the methods for finding the solutions of recurrences. Here we concentrate on the methods proposed by [GKP98] for finding the a closed expression for any x_k. This method shows that $C \geq 9$ is required, which gives a second proof for the optimal ratio 9.

For solving a recurrence as shown in [GKP98] we can perform 4 steps. The general correctness of the method is proved in [GKP98].

A) Closed Form: We bring $\frac{(C-1)}{2}(x_k - x_{k-1}) = x_{k+1}$ with $x_{-1} := 0$ and $x_0 = 1$ into a closed formula that also holds for the starting values. For comparison to [GKP98] we use the notation g instead of x and set $D := \frac{(C-1)}{2}$. We have $g_0 = 0$, $g_1 = 1$ und $g_n = D(g_{n-1} - g_{n-2})$. By $[n = l]$ we denote a serie that has value 1 for $n = l$ and value 0 for all other n. We assume $g_n = 0$ for negative n. Thus a closed formula is given by $g_n = Dg_{n-1} - Dg_{n-2} + 1 \cdot [n = 1]$.

B) Building a power serie with coefficients g_n: We consider the power serie $G(z) := \sum_n g_n z^n$. Inserting the closed form of the preceding paragraph we have:

$$\begin{aligned}
\sum_n g_n z^n &= D \sum_n g_{n-1} z^n - D \sum_n g_{n-2} z^n + \sum_n [n = 1] z^n \\
&= D \sum_n g_n z^{n+1} - D \sum_n g_n z^{n+2} + z \\
&= DzG(z) - Dz^2 G(z) + z.
\end{aligned}$$

C) Closed form for power serie $G(z)$: From (B) we conclude $G(z) = \frac{z}{1 - Dz + Dz^2}$.

D) Developing the power serie $G(z)$: The remaining task is is to make use of (C) for the precise development of the power serie. We will sketch the procedure presented in [GKP98]. In general we have

$G(z) = \frac{P(z)}{Q(z)}$. In our special case we conclude $P(z) = z$ and $Q(z) = 1 - Dz + Dz^2$. By function theory arguments a serie for $G(z)$ with the precise values for g_n is constructed; details are given in [GKP98]. The construction is based on the zeros of $Q(z)$. The main argument is that the zeros of $G(z)$ has to be real in order to obtain reasonable expressions for g_n. The overall conclusion that the zeros of $Q(z)$ are given by $z_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{D}}$. The radicant is negative for $D < 4$ and there are no real-valued solutions in this case. The conclusion is that only $D \geq 4$ (or $C \geq 9$, respectively) guarantees reasonable values for g_n.

The details of calculating g_n precisley from the zeros of $G(z)$ are given in [GKP98]. We would like to present at least the calculations for the recurrence above for $D = 4$. First, $Q(z)$ is expressed by $Q(z) = q_0(1 - p_1 z)^{d_1} \cdots (1 - p_l z)^{d_l}$ where $\frac{1}{p_i}$ is a zero of Q of order d_i. In our example we have $Q(z) = (1 - 2z)^2$ with $q_0 = 1$, $p_1 = 2$ and $d_1 = 2$.

Now the coefficients g_n of $G(z)$ are given by $f_1(n)p_1^n + \cdots f_l(n)p_l^n$ where $f_i(n)$ is a polynomial of degree $d_i - 1$. In our example $f_1(n)$ has degree $d_1 - 1 = 1$ and the coefficient g_n of $G(z)$ is $f_1(n)2^n$.

Additionally, for the polynomial $f_i(n)$ of degree $d_i - 1$, the leading coefficient a_i is presented by

$$a_i = \frac{P\left(\frac{1}{p_i}\right)}{(d_i - 1)!q_0 \prod_{j \neq i}\left(1 - \frac{p_j}{p_i}\right)}.$$

In our special case we have $f_1(n) = a_1 n + a_0$ with $a_1 := \frac{P\left(\frac{1}{2}\right)}{(2-1)!1} = \frac{1}{2}$. Now we have $g_n = \left(\frac{1}{2}n + a_0\right)2^n$. The remaining task is to determine a_0. This can be done by the starting values $g_1 = 1$ or $g_0 = 0$. For $1 = g_1 = \left(\frac{1}{2} + c\right)2 = 1 + 2a_0$ we have $a_0 = 0$, the same holds for $g_0 = 0$. Therfore we conclude $g_n = n2^{n-1}$ which is exactly the above presented solution, which attained equality in any step. The competitive ratio is 9.

Altogether, in this section we have developed different methods for solving discrete motion planning problems by functionals.

3.1.3 2-ray search with bounded distance

Let us assume that in the beginning the maximal distance D from start to target point is given. This means that the rays are bounded by length D. We consider the 2-ray search problem; see Figure 3.5.

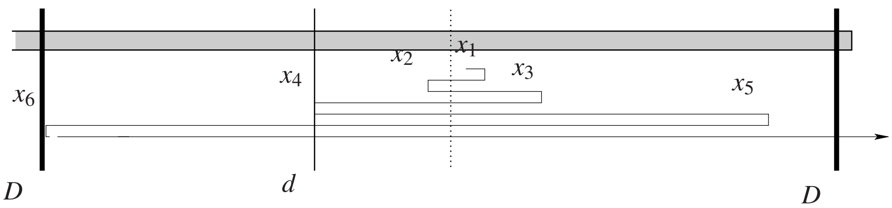

Figure 3.5: Falls wir wissen, dass das Ziel in einer Distanz D liegt, können wir die Strategie optimieren.

If the goal has precisely distance D from the start, the agents runs distance D to one side, back to the start and distance D to the opposite side. In the worst-case this gives a ratio of 3 which is optimal. Now let us assume that the goal is inside an interval $[1, D]$ away from the start, we would like to minimize the ratio $C = 9$.

Obviously, the optimal strategy checks precisely distance $x_k = D$ in the second last step on one side and attains the ratio for the last depth $x_{k-1} < D$ on the opposite side. The step x_{k+1} can be arbitrarily long, it will not decrease the ratio any more; see Figure 3.5. Because of these properties we can also ask for the opposite question. Assume that a ratio $C < 9$ is given. What is the maximal depth D on both side, so that any goal in the distance interval $[1, D]$ away from the start will be found with ratio C.

More precisely, we would like to maintain the ratio $C < 9$ and maximize the second last step x_k. Since $C = 9$ is the overall optimal factor, for $C < 9$ we cannot guarantee factor C for distances up t ∞ or $-\infty$.

In [HIKL99] it was shown, that for $C < 9$ the maximal reach will be attained for a strategy that attains equality in any worst-case step. We can conclude $\frac{\sum_{i=1}^{j+1} x_i}{x_j} = \frac{(C-1)}{2}$ for $j = 0,1,\ldots,k-1$ holds, where $x_0 = 1$ and x_k is the maximal depth.

The choice of $x_0 = 1$ stem from the fact, that the goal is at least one step away from the start. If there is a sub-strategy $(x_1,x_2,\ldots,x_k)$ such that $x_1 < \frac{(C-1)}{2}$ holds. We can simply use a scalar $A > 1$ such that $Ax_1 = \frac{(C-1)}{2}$. The strategy $(Ax_1,Ax_2,\ldots,Ax_k)$ is C-competitive and has larger depth Ax_k. The argumentation that any inequality has to be fulfilled by equality is given in the proof below.

Let us have a look at the results: For $C = 6$ we conclude: $x_1 = 2.5$, $x_2 = 2.5(2.5-1) = 3.75$, $x_3 = 2.5(3.75-2.5) = 3.125 < x_2$, $x_4 = 2.5(x_3 - x_2) < 0$. This means that k is 2 and the strategy attains optimal reach 3.75, the worst-case is attained for $x_0 = 1$ and $x_1 = 2.5$. For $C = 7$ we obtain the strategy: $x_1 = 3$, $x_2 = 3(3-1) = 6$, $x_3 = 3(6-3) = 9$, $x_4 = 3(9-6) = 9$, $x_5 = 3(9-9) < x_3$. We have reach 9 and k equals 4. The worst-case for the ratio is attained at $x_0 = 1$, $x_1 = 3$, $x_2 = 6$ and $x_3 = 9$.

Theorem 3.4 *Let $C < 9$ be the given factor for the 2-ray search problem. For the maximal reach problem there is always an optimal strategy that attains equality in any step.* *[HIKL99]*

Proof. We would like to develop an alternative proof. Let us assume that there is a strategy that attains the maximal reach for given $C < 9$. The goal is at least one step away from the start. We have a sequence $(x_1,x_2,\ldots,x_k)$ such that $\frac{\sum_{i=1}^{j+1} x_i}{x_j} \leq \frac{(C-1)}{2}$ for $j = 0,1,\ldots,k-1$ holds with $x_0 := 1$ and x_k is the maximal depth.

Consider the first index j such that $\frac{\sum_{i=1}^{j+1} x_i}{x_j} < \frac{(C-1)}{2}$ holds. For $j = 0$ we have $x_1 < \frac{(C-1)}{2}$ holds. We can simply use a scalar $A_1 > 1$ such that $A_1 x_1 = \frac{(C-1)}{2}$. The strategy $(A_1x_1,A_1x_2,\ldots,A_1x_k)$ is C-competitive and has larger depth Ax_k. In general let us assume that for $j > 0$ we have $\sum_{i=1}^{j+1} x_i < \frac{(C-1)}{2} x_j$. We enlarge x_{j+1} by factor A_{j+1} such that $\sum_{i=1}^{j} x_i + A_{j+1}x_{j+1} = \frac{(C-1)}{2} x_j$ holds. We exchange the current sequence by $(x_1,x_2,\ldots,x_j,A_{j+1}x_{j+1},A_{j+1}x_{j+2},\ldots)$ and still guarantee $\frac{\sum_{i=1}^{j+1} x'_i}{x'_j} \leq \frac{(C-1)}{2}$ for $j = 0,1,\ldots,k-1$. We have reach $x'_k = A_{j+1}x_k > x_k$ which gives the contradiction. □

Figure 3.6 shows the curve of the function f, where $f(C)$ is the maximal reach that can be attained for C. For any kink in the curve the strategy makes another turn. The number of turns increases. For $C = 7$ we have $k = 3$ turns and for $C = 6$ we have $k = 2$ turns!

The above function f is strictly monotonically increasing in C. This means that by binary search we can easily compute the best ratio C for given reach R. The corresponding reverse function is shown in Figure 3.7.

The above equality-paradigm can also be used for the m-ray search problem, if the given depth is the same on each ray; see [Sch01, Lan00]. For different intervals on the rays up to now no efficient optimization technique is known. Only for some few rays (< 4) a master thesis shows some results; see [Web07].

3.2 Searching for a ray in the plane

In this section, we consider the search for the origin, t, of a ray R in the plane, see Figure 3.8. The searcher has no vision, but recognizes the ray and the ray's origin as soon as the searcher hits the ray. The position of the ray is not known in advance. The searcher moves along a path, Π, starting at a given point, s. Eventually, Π will hit the ray R at a point p and the origin t is detected. The cost of the strategy is given by the length of the path from s to p (i.e., $|\Pi_s^p|$), plus the distance $|pt|$ from p to t. We measure the quality of the path Π for the ray R using the competitive ratio $\frac{|\Pi_s^p|+|pt|}{|st|}$; that is, we compare the length of the searcher's path to the shortest path from s to t. We would like to find a search path Π that guarantees a competitive ratio not greater than C for all possible rays R in the plane. In turn, C should be as small as possible. Similar problems were discussed by Alpern and Gal [AG03]; for example, searching for an unknown line in the plane.

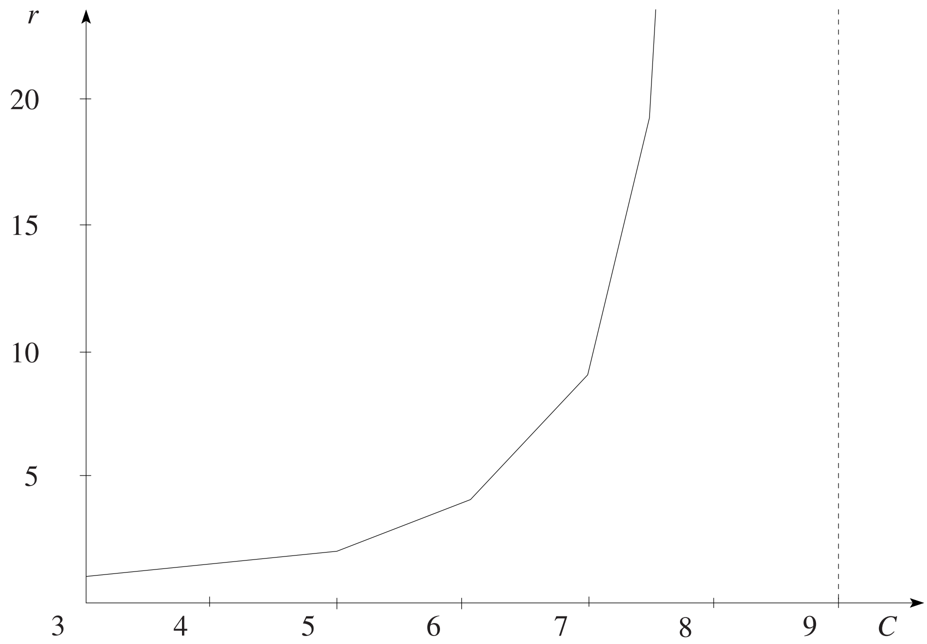

Figure 3.6: Maximal reach depending on the ratio $C < 9$.

3.2.1 The Window Shopper Problem

First, we consider the problem of finding a gift along a shopping window. The agent starts somewhere and looks toward the window. We assume that the item, t, gets into sight if the ray, R, from t to the seachers position, p, is perpendicular to the window. Then the searcher moves toward t.

This problem can be modelled as follows. W.l.o.g. we assume that the line of sight (i.e., the ray, R, we are looking for), is parallel to the X-axis, starts in $(1, y_R)$ for $y_R \geq 0$, and emanates toward the left side of the perpendicular ray R' (the *window*) which starts in $(1,0)$. The searcher starts in the origin $s = (0,0)$; see Figure 3.9. The goal (i.e., the ray's origin t) is discovered as soon as the searcher reaches its height, y_R. After the searcher has discovered the goal, it moves directly to the goal. Note that the shortest distance from s to R' can be fixed to 1 because scaling has no influence on the competitive ratio.

We would like to find a search path, Π, so that for any goal, t, the ratio $\frac{|\Pi_s^p|+|pt|}{|st|} \leq C$ holds, where C is the smallest achievable ratio for all search paths.

Theorem 3.5 *There is a strategy* Π *with an optimal competitive factor of* $1.059\ldots$ *for searching the origin of a ray, R, that emanates from a known ray R' perpendicular to R.* *[EFK+06]*

Proof. Apparently a good search path moves simultaneously along and towards the wall; that is, in positive X- and Y-direction. Note that the competitive factor for any reasonable strategy converges to 1 for goals with very small Y-coordinate and also for goals with a large Y-coordinate. Therefore, the first part of our path, Π, is a line segment up to a point (a,b). The second part is a curve, $f(x)$, that converges to the wall and maintains the competitive factor that was achieved by the line segment in the first part of the search; see Figure 3.9.

Thus, we solve two tasks.

1. We will design a search path Π that consists of the following three parts (or conditions); see Figure 3.10(i).

Π_1: A straight line segment from $(0,0)$ to some point (a,b) where the competitive ratio strictly increases from $C = 1$ to $C_{\max}$ for goals from $(1,0)$ to $(1,b)$.

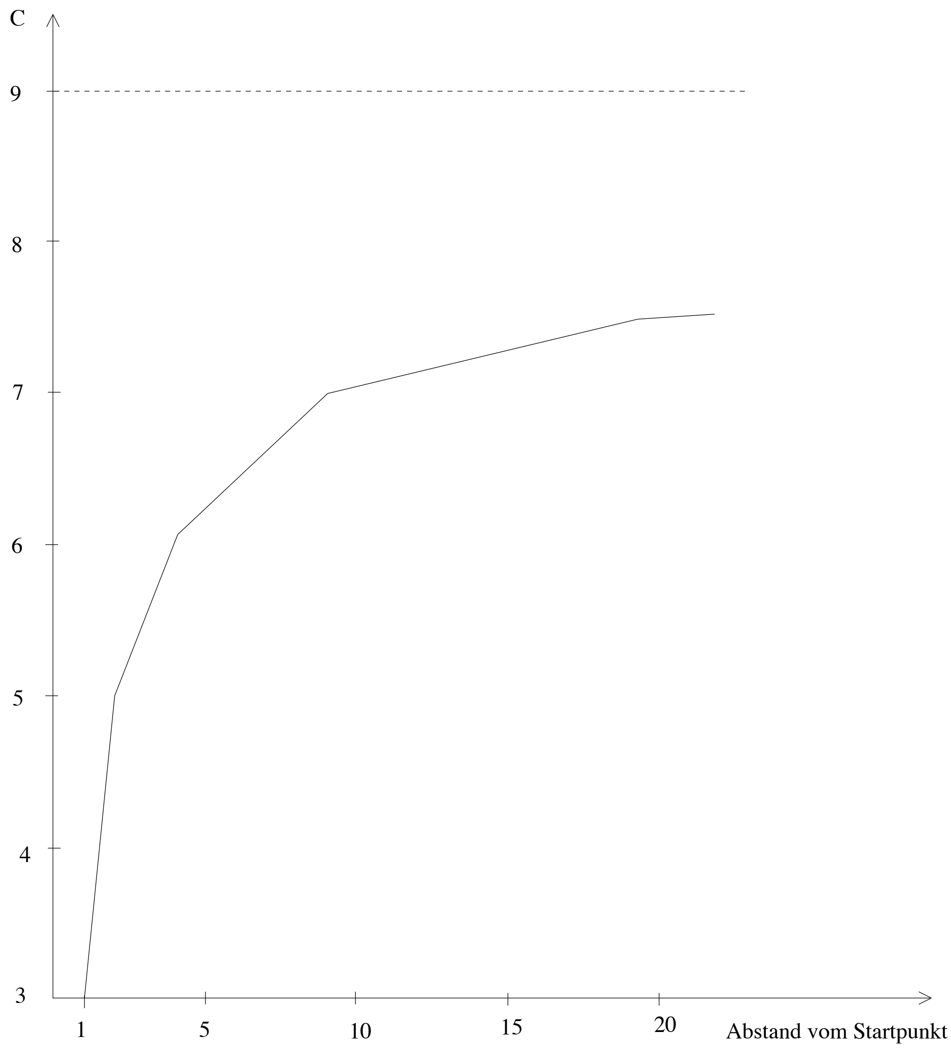

Figure 3.7: Optimal competitive ratio for given reach.

Π_2: A strictly monotone curve f from (a,b) to some point $(1,D)$ on R' where the competitive ratio is exactly $C_{\max}$ for all goals from $(1,b)$ to $(1,D)$.

Π_3: A ray starting from $(1,D)$ to $(1,\infty)$ where the competitive ratio strictly decreases from $C_{\max}$ to 1 for goals from $(1,D)$ to $(1,\infty)$.

Furthermore, we prove that the full path Π is convex. The competitive ratio of Π is $C_{\max}$.

2. We will show that such a path is optimal and the best achievable ratio is $C_{\max}$.

We start with the second task. Let us assume that we have designed a search path Π with the given properties and let us assume that there is an optimal search path K with $K \neq \Pi$, see Figure 3.10(ii).

The path K might hit the ray B from $(1,b)$ to $(-\infty,b)$ at a point p_1 to the left of (a,b). Then the ratio $\frac{|K_s^{p_1}|+|p_1(1,b)|}{|s(1,b)|}$ is bigger than $C_{\max} = \frac{|s(a,b)|+|(a,b)(1,b)|}{|s(1,b)|}$. On the other hand, K might move to the right of (a,b) and hits Π_2 at a point p_2 between B and the ray D from $(1,D)$ to $(-\infty,D)$. In this case, the length of $K_s^{p_2}$ has to be bigger than $|\Pi_s^{p_2}|$ because Π is fully convex. Thus, the ratio $\frac{|K_s^{p_2}|+|p_2(1,p_{2_y})|}{|s(1,p_{2_y})|}$ is bigger than $C_{\max} = \frac{|\Pi_s^{p_2}|+|p_2(1,p_{2_y})|}{|s(1,p_{2_y})|}$, where p_{2_y} denotes the Y-coordinate of p_2. This also holds if K hits R' first and p_2 is equal to $(1,D)$; see the dotted path in Figure 3.10(ii).

This means that K has to follow Π from s up to some point beyond B and might leave Π_2 then. In this case K has at least the ratio $C_{\max}$ and Π is optimal, too.

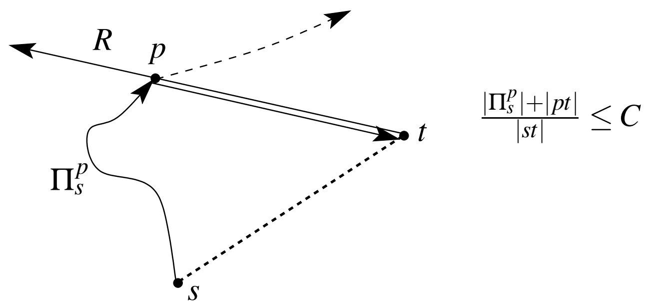

Figure 3.8: Die Suche nach dem Ursprung t eines Strahles R.

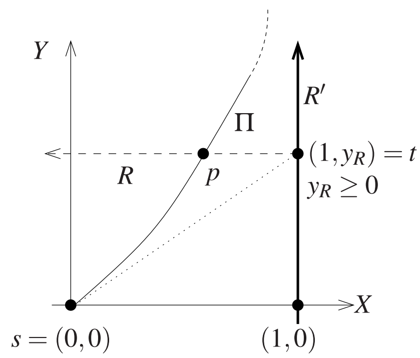

Figure 3.9: A strategy for the window shopper!

It remains to show that we can design a path with the given properties. As already mentioned, the motivation for the construction is the following: In the very beginning the ratio starts from 1 and has to increase for a while, this is true for any strategy. Additionally, any reasonable strategy should be monotone in x and y. Moving backwards or away from the window will allow shortcuts with a smaller ratio. Therefore it is reasonable that we will get closer and closer to the window R' and the factor should decrease to 1. So, finally, we can hit R' because at the end the ratio will not be the worst case. Furthermore, in many applications strategies are designed by the fact that they achieve exactly the same factor for a set of goals. Altogether, we would like to design a path Π by the properties formulated above, and — as we already know — such a strategy is optimal.

With the first two conditions for Π_1 and Π_2 we determine a and b. We consider the line segment from the origin $(0,0)$ to (a,b) with $a,b > 0$ to be parametrized by (ta,tb) for $t \in [0,1]$. The competitive factor for Π_1 is given by

$$C(t) = \frac{t\sqrt{a^2+b^2}+1-ta}{\sqrt{1+t^2b^2}}, \quad t \in [0,1].$$

We want $C(t)$ to be a monotone and increasing function. From $C'(t) \geq 0\ \forall t \in [0,1]$ we conclude

$$\begin{aligned}
C'(t) &= \frac{(\sqrt{a^2+b^2}-a)(1+t^2b^2)-(t(\sqrt{a^2+b^2}-a)+1)tb^2}{\sqrt{1+t^2b^2}(1+t^2b^2)} \geq 0 \quad \forall t \in [0,1] \\
&\Leftrightarrow \sqrt{a^2+b^2}-a \geq tb^2 \quad \forall t \in [0,1] \\
&\Leftrightarrow \sqrt{a^2+b^2}-a \geq b^2 \\
&\Leftrightarrow a^2+b^2 \geq b^4+2ab^2+a^2 \\
&\Leftrightarrow 1-2a \geq b^2 .
\end{aligned}$$

Hence, $a \leq \frac{1-b^2}{2}$ follows. From now on we set $a := \frac{1-b^2}{2}$. For $t = 1$ and $a := \frac{1-b^2}{2}$ we obtain a competitive

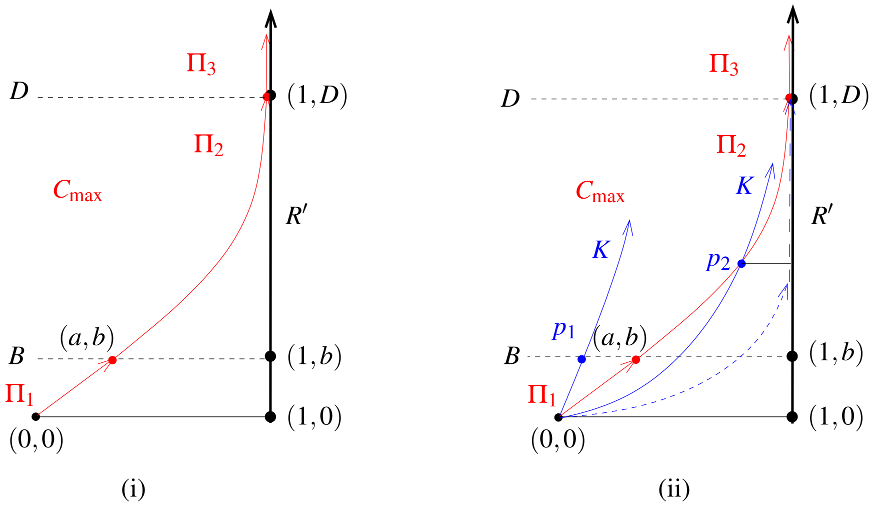

Figure 3.10: An arbitrary search path K is not better than Π.

factor of

$$\begin{aligned}\frac{\sqrt{a^2+b^2}+1-a}{\sqrt{1+b^2}} &= \frac{\sqrt{(\frac{1-b^2}{2})^2+b^2}+1-\frac{1-b^2}{2}}{\sqrt{1+b^2}} = \frac{\sqrt{\frac{1-2b^2+b^4+4b^2}{4}}+\frac{1}{2}+\frac{b^2}{2}}{\sqrt{1+b^2}} \\ &= \frac{\sqrt{(\frac{1+b^2}{2})^2}+\frac{1}{2}(1+b^2)}{\sqrt{1+b^2}} = \sqrt{1+b^2} =: C\,. \end{aligned} \tag{3.4}$$

We can consider the line segment Π_1 also as a function of $x \in [0,a]$. Now, C is the worst case competitive factor for $x \in [0,a]$ and goals t between $[1,0]$ and $[1,b]$.

For Π_2 we construct a curve $f(x)$ for $x \in [a,1]$ that runs from $[a,b]$ to some point $[1,D]$ and achieves the ratio $C = \sqrt{1+b^2}$ for all goals t between $[1,b]$ and $[1,D]$. This means that the length of the path of the searcher (i.e., the line segment up to (a,b), the part of the curve f up to the height y_R, and the final line segment to the goal $(1,y_R)$) is equal to C times the Euclidean distance from the origin $(0,0)$ to the goal $(1,y_R)$. Thus, f can be defined by the differential equation

$$\sqrt{a^2+b^2}+1-x+\int_a^x \sqrt{1+f'(t)^2}\,dt = C\cdot\sqrt{1+f(x)^2}. \tag{3.5}$$

We would like to rearrange equation (3.5) in order to apply standard methods for solving differential equations. Derivating equation (3.5) and squaring twice gives

$$\begin{aligned} & \sqrt{1+f'(x)^2}-1 = \frac{C}{2}\cdot\frac{1}{\sqrt{1+f(x)^2}}\cdot 2f(x)f'(x) \\ \Leftrightarrow\quad & 1+f'(x)^2-2\sqrt{1+f'(x)^2}+1 = C^2\frac{f(x)^2f'(x)^2}{1+f(x)^2} \\ \Leftrightarrow\quad & f'(x)^2\left[1-C^2\frac{f(x)^2}{1+f(x)^2}\right]+2 = 2\sqrt{1+f'(x)^2} \\ \Leftrightarrow\quad & f'(x)^4\left[1-C^2\frac{f(x)^2}{1+f(x)^2}\right]^2+4f'(x)^2\left[1-C^2\frac{f(x)^2}{1+f(x)^2}\right] = 4f'(x)^2\,. \end{aligned}$$

The curve f was assumed to be strictly monotone, which means $f'(x)\neq 0$. Therefore we have

$$\Leftrightarrow\quad f'(x)^2\left[1-C^2\frac{f(x)^2}{1+f(x)^2}\right]^2 = 4C^2\frac{f(x)^2}{1+f(x)^2}$$

$$
\begin{aligned}
&\Leftrightarrow \quad f'(x)^2 = \left[\frac{1+f(x)^2}{1+(1-C^2)f(x)^2}\right]^2 4C^2 \frac{f(x)^2}{1+f(x)^2} \\
&\Leftrightarrow \quad f'(x)^2 = 4C^2 \frac{(1+f(x)^2)f(x)^2}{\left(1+(1-C^2)f(x)^2\right)^2} \\
&\Leftrightarrow \quad f'(x) = 2C\frac{\sqrt{1+f(x)^2}f(x)}{1+(1-C^2)f(x)^2}\,.
\end{aligned} \tag{3.6}
$$

Note that the point $(a,b) = (\frac{1-b^2}{2},b)$ lies on f and C is equal to $\sqrt{1+b^2}$. Altogether, we have to solve the differential equation

$$
y' = 1\cdot 2\sqrt{1+b^2}\frac{\sqrt{1+y^2}y}{1-b^2y^2} = 1\cdot g(y) \tag{3.7}
$$

for $y = f(x)$ with starting point $(\frac{1-b^2}{2},b)$.

Equation (3.7) is a first order differential equation $y' = h(x)g(y)$ with separated variables and point (k,l) on y. A general solution is given by

$$
\int_l^y \frac{dt}{g(t)} = \int_k^x h(z)dz;
$$

see Walter [Wal86]. Thus, we have to solve

$$
\int_b^y \frac{1-b^2t^2}{2\sqrt{1+b^2}\sqrt{1+t^2}t}dt = \int_{(1-b^2)/2}^x 1\cdot dz = x-(1-b^2)/2
$$

By simple analysis, we obtain

$$
x = -\frac{b^2\sqrt{1+y^2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{1+y^2}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+b^2}}\right) - \sqrt{1+b^2}}{2\sqrt{1+b^2}}
$$

which is the solution for the inverse function $x = f^{-1}(y)$. By simple analysis we get

$$
x' = \frac{1}{g(y)} = -\frac{(b^2y^2-1)}{2\sqrt{1+y^2}y\sqrt{(1+b^2)}} \geq 0 \text{ for } y \in [0,1/b]
$$

and

$$
x'' = -\frac{(b^2y^2+2y^2+1)}{2(1+y^2)^{3/2}\sqrt{1+b^2y^2}} \leq 0 \text{ for } y \geq 0,.
$$

Because $x = f^{-1}(y)$ is concave in the given interval, $y = f(x)$ is convex. Additionally, f^{-1} attains a maximum for $y = \frac{1}{b}$.

Altogether we have a situation for the inverse function $x = f^{-1}(y)$ for $y \in \left[0,\frac{1}{b}\right]$ as shown in Figure 3.11.

Now, we have to find a value for b so that $f^{-1}(\frac{1}{b})$ is equal to 1, so that f^{-1} behaves as depicted in Figure 3.11(ii). That is, we have to find a solution for

$$
1 = -\frac{b^2\sqrt{1+\frac{1}{b^2}} + \operatorname{arctanh}\left(\frac{1}{\sqrt{1+\frac{1}{b^2}}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+b^2}}\right) - \sqrt{1+b^2}}{2\sqrt{1+b^2}}\,. \tag{3.8}
$$

This fixes b and, in turn, D to $\frac{1}{b}$. Note that in this case $y = f(x)$ has the desired properties for $x \in [a,1] = \left[\frac{1-b^2}{2},1\right]$.

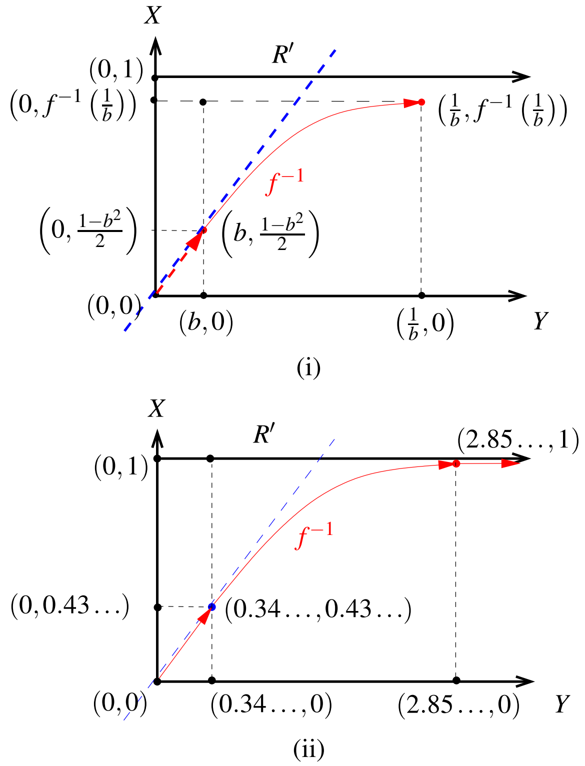

Figure 3.11: The inverse situation of the window shopper problem. The curve f^{-1} should hit the line $X = 1$.

We have already seen that $y = f(x)$ is convex for $x \in [a,1]$. Additionally, the line segment from $(0,0)$ to (a,b) is convex. To show that the conjunction of both elements is also convex, we have to show that the tangent to f at (a,b) is equal to a prolongation of the line segment; see Figure 3.11. In other words, we have to show $f^{-1\prime}(b) = \frac{a}{b} = \frac{1-b^2}{2b}$. This is equivalent to $\frac{1}{g(b)} = \frac{1-b^2}{2b}$ which is obviously true.

By solving equation (3.8) numerically, we get $b = 0.34\ldots$ This gives $D = \frac{1}{b} = 2.859\ldots$, $a = \frac{1-b^2}{2} = 0.43\ldots$ and a worst-case ratio $C = \sqrt{1+b^2} = 1.05948\ldots$ The corresponding curve f^{-1} is shown in Figure Figure 3.11(ii).

Altogether, by combining Π_1 (the line segment), Π_2 (the constructed curve f), and Π_3 (the ray from $(1,D)$ to $(1,\infty)$), we obtain a convex curve with the given properties and an optimal competitive factor of $C = \sqrt{1+b^2} = 1.05948\ldots$ □

3.2.2 General rays in the plane

Now we turn over to arbitrary rays in the plane. We will first show that a *logarithmic spiral* is an appropriate competitive strategy, finally we will construct a lower bound. A logarithmic spiral is defined in polar coordinates by $(\varphi, d \cdot e^{\varphi \cot(\alpha)})$ for $d > 0$ and $-\infty < \varphi < \infty$ see Figure 3.12 for an example. Note, that we can scale so that $d = 1$.

A logarithmic spiral has some nice properties. The center point of the spiral is given by the origin $s = (0,0)$. The angle α expresses the excentricity of the spiral. For every point p on the spiral the line through p and s and the tangent T_p on p build the same angle α. For $\alpha = \pi/2$ the spirals degenerates to a circle. For two points a and b on the spiral, the length of the spiral between a and b is given by $\frac{1}{|\cos\alpha|}(|bs| - |as|)$ for $|as| > |bs|$. This means that the length of the spiral from the center to some point b is given by $\frac{1}{|\cos\alpha|}|bs|$, for details see also [BSMM00].

The spiral expands successively and will finally hit every ray in the plane. Obviously, the worst case is attained for a tangent to the spiral. In the following we denote a spiral path by Π and the corresponding ratio by C_Π.

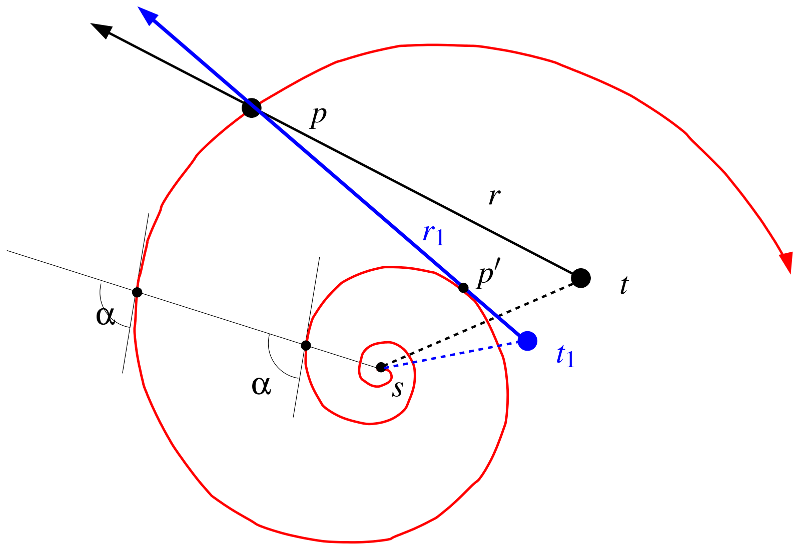

Figure 3.12: A logarithmic spiral is defined by an angle α. A tangent to the spiral will maximize the ratio.

Lemma 3.6 *Given a logarithmic spiral Π, the ray that maximizes the ratio C_Π is a tangent T to the spiral.*

Proof. Consider a ray r emanating from the point t, and the first intersection p with the spiral; see Figure 3.12. We can increase the ratio C_Π by rotating r counterclockwise around p until the ray is almost a tangent to the spiral. Additionally, t_1 gets closer to s. Note, that p' in Figure 3.12 is not actually an intersection, but the searcher moving on the spiral slightly misses the ray r_1 in p', but detects the ray in p. However, p' is arbitrarily close to the spiral; thus, we consider p' to be a point on the spiral. We call p' *tangent point*. □

Now we will proceed as follows, we will compute an optimal spiral Π given by an optimal angle α for the *orthogonal points*, q', on the tangent T_q, see Figure 3.13. Fortunately, the given ratio $C_\Pi(\alpha)$ is the same for all tangents! Afterwards, an adversary strategy can move the starting point of the ray along the tangent in order to maximize the ratio, see Figure 3.13. This means that the adversary strategy can choose the angle β.

By the law of sine we have $|st| = \frac{|sq'|}{\sin\beta}$ and $\frac{|tq'|}{\cos\beta} = |ts|$. Now we have $\frac{|\pi_s^p|+|pq'|}{|sq'|} = C_\Pi(\alpha)$ (the ratio for q') and $\frac{|\pi_s^p|+|pq'|+|q't|}{|st|} =: C_\Pi(\alpha,\beta)$ (the ratio for t). Substituting the above dependencies we have: $C_\Pi(\alpha,\beta) = C_\Pi(\alpha)\sin(\beta)+\cos(\beta)$. Altogether, we first will minimize $C_\Pi(\alpha)$ over α and then we maximize $C_\Pi(\alpha,\beta)$ over β.

Lemma 3.7 *We can minimize the ratio for the closest point, q', on a tangent T_q by choosing an optimal angle α. If we attain a ratio $C_\Pi(\alpha)$ for the orgin q', an adversary can move the starting point along the tangent which is determined by an angle β. The ratio will be given by $C_\Pi(\alpha,\beta) := C_\Pi(\alpha)\sin(\beta)+\cos(\beta)$.*

Note, that it makes no sense for the adversary to move the starting point to the *right* of q', the ratio will obviously decrease.

Now, we would like to compute the distance $|qq'|$. In the following we consider $\alpha < \pi/2$ and the spiral turns counterclockwise, see Figure 3.14. This has the advantage that the angles are positive in the mathematical sense. Let $\gamma(\alpha)$ denote the angle between sq and sp, see Figure 3.14. For $q := (\varphi_q, e^{\varphi_q \cot\alpha})$, we have $p := (\varphi_q + 2\pi + \gamma(\alpha), e^{(\varphi_q+2\pi+\gamma(\alpha))\cot\alpha})$. The angle $\gamma(\alpha)$ can be determined by an equation. The proof of Lemma 3.8 is a simple exercise.

Note, that a line running through $q' = (\varphi_{q'}, r_{q'})$ and perpendicular to a line with angle $\varphi_{q'}$ is given in polar coordinates by $(\varphi, \frac{r_{q'}}{\cos(\varphi-\varphi_{q'})})$. In our case the tangent T_q is perpendicular to the line given by $\varphi_{q'}$ and runs through q'. In turn the angle $\varphi_{q'}$ is given by $\varphi_q - \pi/2 + \alpha$.

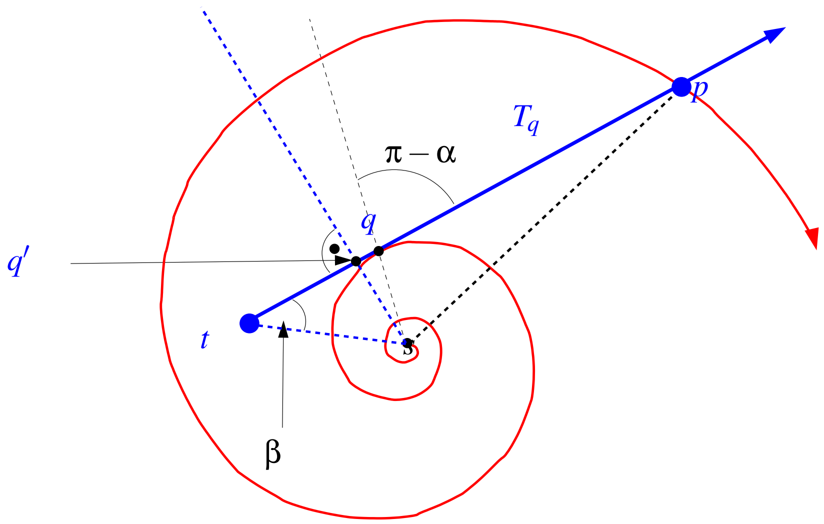

Figure 3.13: We would like to optimize the spiral for the closest point, q', from s on a tangent T_q.

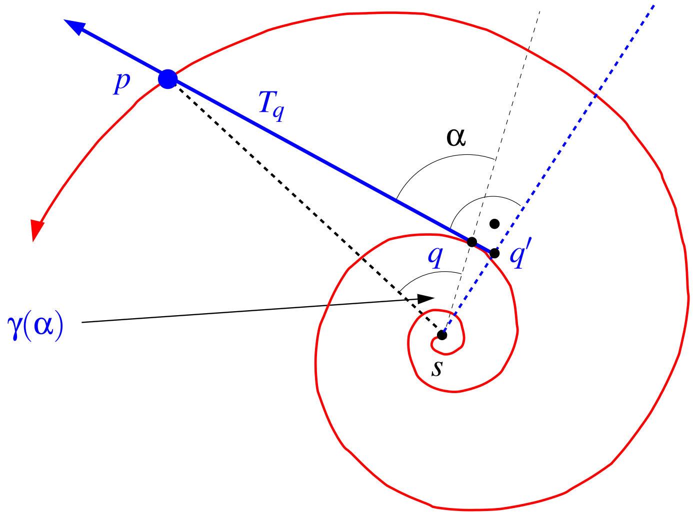

Figure 3.14: Let q' be the point on T_q with shortest distance to s. If the angle $\gamma(\alpha)$ in $\triangle qsp$ is given, we can determine the ratio for q'.

Lemma 3.8 *The angle $\gamma(\alpha) := \angle qsp$ is given by the solution to*

$$\frac{\sin\alpha}{\sin(\alpha - \gamma(\alpha))} = \mathcal{E}^{\cot\alpha(2\pi+\gamma(\alpha))}.$$

Lemma 3.8 gives us a formula for computing $\gamma(\alpha)$ at least numerically for every α. Therefore we will be able to compute the best spiral for a tangent T_q on q.

Theorem 3.9 *Given a spiral and a tangent, T_q, to the spiral. Let q' be the closest point to s on T_q. The ratio $C_\Pi(q')$ for q' on T_q depends only on the spiral parameter α and is given by*

$$C_{q'}(\alpha) = \frac{\mathcal{E}^{\cot\alpha(2\pi+\gamma(\alpha)))}}{\sin\alpha\cdot\cos\alpha} + \frac{\mathcal{E}^{b(2\pi+\gamma(\alpha))}\cdot\sin(\gamma(\alpha))}{\sin^2\alpha} + \cot\alpha\,.$$

Its minimum value is $22.49084026\ldots$ *for* $\cot\alpha_{opt} = 0.11371\ldots$ *or* $\alpha_{opt} := 1.4575\ldots$

Proof. Consider the triangle $\triangle psq$, see Figure 3.14. Because q is a point on the spiral we have $|sq| = \mathcal{E}^{\cot\alpha(\theta_q)}$ for some θ_q. Additionally, we have $\theta_p = \theta_q + 2\pi + \gamma(\alpha)$. Further, we have $\angle sqp = \pi - \alpha$. Applying the law of sine yields

$$\frac{|sp|}{\sin(\pi-\alpha)} = \frac{|sp|}{\sin\alpha} = \frac{|pq|}{\sin(\gamma(\alpha))} \Leftrightarrow |pq| = \frac{\mathcal{E}^{\cot\alpha(\theta_p)}\sin\gamma(\alpha)}{\sin\alpha} = \frac{\mathcal{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))}\sin(\gamma(\alpha))}{\sin\alpha}.$$

Because the triangle $\triangle sq'q$ is right angled, we have $|qq'| = |sq|\cos\alpha = \mathcal{E}^{\cot\alpha(\theta_q)}\cos\alpha$; thus, the distance $|pq'| = |pq| + |qq'|$ is given as

$$|pq'| = \frac{\mathcal{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))}\sin(\gamma(\alpha))}{\sin\alpha} + \mathcal{E}^{\cot\alpha(\theta_q)}\cos\alpha\,.$$

The length of the arc Π_s^p on the spiral from s to p is given by $|\Pi_s^p| = \frac{\mathcal{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))}}{\cos\alpha}$, Now, using $|sq'| = |sq|\sin\alpha = \mathcal{E}^{\cot\alpha(\theta_q)}\sin\alpha$, we can compute the ratio $C_{q'}(\alpha)$:

$$\begin{aligned}
C_{q'}(\alpha) &= \frac{|\Pi_s^p| + |pq'|}{|sq'|} \\
&= \frac{\frac{1}{\cos\alpha}\mathcal{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))} + \frac{1}{\sin\alpha}\mathcal{E}^{\cot\alpha(\theta_q+2\pi+\gamma(\alpha))}\sin\gamma(\alpha) + \mathcal{E}^{\cot\alpha\theta_q}\cos\alpha}{\mathcal{E}^{\cot\alpha\theta_q}\sin\alpha} \\
&= \frac{\frac{1}{\cos\alpha}\mathcal{E}^{\cot\alpha(2\pi+\gamma(\alpha))} + \frac{1}{\sin\alpha}\mathcal{E}^{\cot\alpha(2\pi+\gamma(\alpha))}\sin\gamma(\alpha) + \cos\alpha}{\sin\alpha} \\
&= \left(\frac{1}{\sin\alpha\cdot\cos\alpha} + \frac{\sin\gamma(\alpha)}{\sin^2\alpha}\right)\mathcal{E}^{\cot\alpha(2\pi+\gamma(\alpha))} + \cot\alpha\,.
\end{aligned}$$

We observe that $C_{q'}(\alpha)$ is independent of θ_q, that is, the value of $C_\Pi(q')$ is the same for every given tangent T. Now, the searcher is allowed to minimize the search costs by choosing an appropriate value for α. Evaluating $C_{q'}(\alpha)$ numerically yields a minimum value of $22.49084026\ldots$ for $\cot\alpha = 0.11371\ldots$ or $\alpha_{\mathrm{opt}} := 1.4575\ldots$ □

Finally, an adversary is allowed to choose a starting point, t, along the tangent T_q. By Lemma 3.7 we have to choose β so that $C_\Pi(\alpha_{\mathrm{opt}},\beta) = C(\alpha_{\mathrm{opt}})\sin\beta + \cos\beta$ is maximal. Therefore we have to find a solution for $C(\alpha_{\mathrm{opt}})\cos\beta - \sin\beta$ (first derivative in β) which gives $\beta_{\mathrm{worst}} := 1.526363\ldots$ and $C_\Pi(\alpha_{\mathrm{opt}},\beta_{\mathrm{worst}}) = 22.51306056\ldots$

Corollary 3.10 *The spiral strategy with $\alpha_{opt} := 1.4575\ldots$ is optimal among all spirals and obtains a competitive factor of $C_\Pi(\alpha_{opt},\beta_{worst}) = 22.51306056\ldots$ for angle $\beta_{worst} := 1.526363\ldots$*

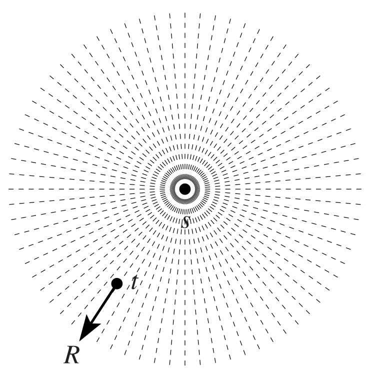

Figure 3.15: A ray, R, that emanates from t and is part of a ray that emanates from s.

Finally, we are interested in a lower bound. To get a lower bound on the competitive ratio for our problem, we discuss the following subproblem: We require that the ray, R, we are looking for is part of a rays that emanates from the searcher's start point, s (i.e., the start point, s, lies on the the extension of R to a straight line)

If we consider the full bundle of lines passing through s, the given problem is equivalent to the problem of searching for a point in the plane as presented by Alpern and Gal [AG03]. We assume that the searcher detects the goal if it is swept by the radius vector of its trajectory; that is, the searchers knows the position of the goal as soon as it hits the ray emanating from the goal. Alpern and Gal [AG03] showed that among all monotone and periodic strategies, a logarithmic spiral represented by polar coordinates $(\theta, \mathcal{E}^{b\theta})$ gives the best search strategy in this setting. A strategy S represented by its radius vector $X(\theta)$ is called periodic and monotone if θ is always increasing and X also satisfies $X(\theta+2\pi) \geq X(\theta)$.

The factor of the best achievable monotone and periodic strategy is given by $17.289\ldots$, see Alpern and Gal [AG03]. Note, that the searcher does not have to reach the ray's origin in this setting.

Unfortunately, it was not shown that a periodic and monotone strategy is the best strategy for this problem. Alpern and Gal state that it *seems to be a complicated task* to show that the spiral optimizes the competitive factor. Thus, the given factor cannot be adapted to be a lower bound to our problem. Therefore, we consider a discrete bundle of n rays that emanate from the start and which are separated by an angle $\alpha = \frac{2\pi}{n}$, see Figure 3.16. We are searching for a goal on one of the n rays.[1] Again, the goal is detected if it is swept by the radius vector of the trajectory. Note that if n goes to infinity we are back to the original problem. But we can neither assume that we have to visit the rays in a periodic order nor that the depth of the visits increases in every step. Thus, we represent a search strategy, S, as follows: In the kth step, the searcher hits a ray—say ray i—at distance x_k from the origin, moves a distance $\beta_k x_k - x_k$ along the ray i, and leaves the ray at distance $\beta_k x_k$ with $\beta_k \geq 1$. Then, it moves to the next ray within distance $\sqrt{(\beta_k x_k)^2 - 2\beta_k x_k x_{k+1} \cos\gamma_{i,i+1} + x_{k+1}^2}$, see Figure 3.16. Note, that any search strategy for our problem can be described in this way.

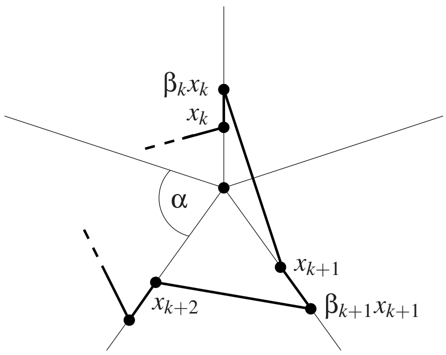

Figure 3.16: A bundle of n rays and the representation of a strategy.

Let us assume that the ray i is visited the next time at index J_k. The worst case occurs if the searcher slightly misses the goal while visiting ray i up to distance x_k. Instead, it finds the goal at step x_{J_k} on ray i arbitrarily close to $\beta_k x_k$. Either we have $x_{J_k} > \beta_k x_k$; that is, the searcher discovers the goal in distance x_{J_k} on ray i and moves $x_{J_k} - \beta_k x_k$ to the goal, or we have $x_{J_k} < \beta_k x_k$. In the latter case, the searcher moves $\beta_k x_k - x_{J_k}$ from x_{J_k} and finds the goal by accident. In both cases, the searcher moves $|x_{J_k} - \beta_k x_k|$ in the last step. Altogether, the competitive factor, $C(S)$, is bigger than

$$\frac{|x_{J_k} - \beta_k x_k| + \sum_{i=1}^{J_k-1} \beta_i x_i - x_i + \sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos\gamma_{i,i+1} + x_{i+1}^2}}{\beta_k x_k}.$$

By simple trigonometry, the shortest distance from $\beta_i x_i$ to a neighboring ray is given by $\beta_i x_i \sin\frac{2\pi}{n}$. Fortunately, this distance is smaller than the distance

$$\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos\gamma_{i,i+1} + x_{i+1}^2}$$

[1] Note that the searcher is not confined to walk on the rays, but can move arbitrarily in the plane; in contrast to the m-ray search problem.

to any other ray. Thus, we have

$$C(S) > \frac{\sum_{i=1}^{J_k-1} \beta_i x_i}{\beta_k x_k} \sin \frac{2\pi}{n} .$$

Altogether, we have to find a lower bound for $\frac{\sum_{i=1}^{J_k-1} f_i}{f_k}$, where J_k denotes the index of the next visit of the ray of x_k and $f_i = \beta_i x_i$ denotes the search depth in step i. Fortunately, this problem is the same problem as in the competitive analysis for the usual m-ray problem where the searcher can move only along the rays. It was shown in Lemma 3.3 (see also Gal [Gal80] and Baeza-Yates et al. [BYCR93]) that for this problem there is an optimal strategy that visits the rays with increasing depth and in a periodic order; that is, $J_k = k+n$ and $i = k$. Applying Theorem 3.2 the best achievable strategy is given by $f_i = (n/(n-1))^i$. Altogether, this results in a function

$$(n-1)\left(\frac{n}{n-1}\right)^n \sin \frac{2\pi}{n}$$

for n rays. We can make n arbitrarily big because our construction is valid for every n. Note that we also have a lower bound for the problem of searching a point in the plane; this lower bound is close to the factor that is achieved by a spiral search.

Theorem 3.11 *For the ray search problem there is no strategy that achieves a better factor than*

$$\lim_{n\to\infty} (n-1)\left(\frac{n}{n-1}\right)^n \sin \frac{2\pi}{n} = 17.079\ldots$$

Additionally, every strategy for searching a point in the plane achieves a competitive factor bigger then $17.079\ldots$ *(the optimal spiral achieves a factor of* $17.289\ldots$ *[Gal80]).*

Bibliography

[Ad80] H. Abelson and A. A. diSessa. *Turtle Geometry*. MIT Press, Cambridge, 1980.

[AFM00] E. M. Arkin, S. P. Fekete, and J. S. B. Mitchell. Approximation algorithms for lawn mowing and milling. *Comput. Geom. Theory Appl.*, 17:25–50, 2000.

[AG03] Steve Alpern and Shmuel Gal. *The Theory of Search Games and Rendezvous*. Kluwer Academic Publications, 2003.

[AKS02] Susanne Albers, Klaus Kursawe, and Sven Schuierer. Exploring unknown environments with obstacles. *Algorithmica*, 32:123–143, 2002.

[BRS94] Margrit Betke, Ronald L. Rivest, and Mona Singh. Piecemeal learning of an unknown environment. Technical Report A.I. Memo No. 1474, Massachusetts Institute of Technology, March 1994.

[BSMM00] Ilja N. Bronstein, Konstantin A. Semendjajew, Gerhard Musiol, and Heiner Mühlig. *Taschenbuch der Mathematik*. Verlag Harry Deutsch, Frankfurt am Main, 5th edition, 2000.

[BYCR93] R. Baeza-Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Inform. Comput.*, 106:234–252, 1993.

[DJMW91] G. Dudek, M. Jenkin, E. Milios, and D. Wilkes. Robotic exploration as graph construction. *Transactions on Robotics and Automation*, 7:859–865, 1991.

[DKK01] Christian A. Duncan, Stephen G. Kobourov, and V. S. Anil Kumar. Optimal constrained graph exploration. In *Proc. 12th ACM-SIAM Symp. Discr. Algo.*, pages 307–314, 2001.

[DKK06] Christian A. Duncan, Stephen G. Kobourov, and V. S. Anil Kumar. Optimal constrained graph exploration. *ACM Trans. Algor.*, 2:380–402, 2006.

[EFK+06] Andrea Eubeler, Rudolf Fleischer, Tom Kamphans, Rolf Klein, Elmar Langetepe, and Gerhard Trippen. Competitive online searching for a ray in the plane. In Sándor Fekete, Rudolf Fleischer, Rolf Klein, and Alejandro López-Ortiz, editors, *Robot Navigation*, number 06421 in Dagstuhl Seminar Proceedings, 2006.

[Gal80] Shmuel Gal. *Search Games*, volume 149 of *Mathematics in Science and Engeneering*. Academic Press, New York, 1980.

[GKP98] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics*. Addison-Wesley, 1998.

[GR03] Yoav Gabriely and Elon Rimon. Competitive on-line coverage of grid environments by a mobile robot. *Comput. Geom. Theory Appl.*, 24:197–224, 2003.

[HIKL99] Christoph Hipke, Christian Icking, Rolf Klein, and Elmar Langetepe. How to find a point on a line within a fixed distance. *Discrete Appl. Math.*, 93:67–73, 1999.

[IKKL00a] Christian Icking, Thomas Kamphans, Rolf Klein, and Elmar Langetepe. Exploring an unknown cellular environment. In *Abstracts 16th European Workshop Comput. Geom.*, pages 140–143. Ben-Gurion University of the Negev, 2000.

[IKKL00b] Christian Icking, Thomas Kamphans, Rolf Klein, and Elmar Langetepe. Exploring an unknown cellular environment. Unpublished Manuscript, FernUniversität Hagen, 2000.

[IKKL05] Christian Icking, Tom Kamphans, Rolf Klein, and Elmar Langetepe. Exploring simple grid polygons. In *11th Internat. Comput. Combin. Conf.*, volume 3595 of *Lecture Notes Comput. Sci.*, pages 524–533. Springer, 2005.

[IPS82] A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM J. Comput.*, 11:676–686, 1982.

[KL03] Tom Kamphans and Elmar Langetepe. The Pledge algorithm reconsidered under errors in sensors and motion. In *Proc. of the 1th Workshop on Approximation and Online Algorithms*, volume 2909 of *Lecture Notes Comput. Sci.*, pages 165–178. Springer, 2003.

[Kle97] Rolf Klein. *Algorithmische Geometrie*. Addison-Wesley, Bonn, 1997.

[Lan00] Elmar Langetepe. *Design and Analysis of Strategies for Autonomous Systems in Motion Planning*. PhD thesis, Department of Computer Science, FernUniversität Hagen, 2000.

[LB99] Sharon Laubach and Joel Burdick. RoverBug: Long range navigation for mars rovers. In Peter Corke and James Trevelyan, editors, *Proc. 6th Int. Symp. Experimental Robotics*, volume 250 of *Lecture Notes in Control and Information Sciences*, pages 339–348. Springer, 1999.

[Lee61] C. Y. Lee. An algorithm for path connections and its application. *IRE Trans. on Electronic Computers*, EC-10:346–365, 1961.

[LS87] V. J. Lumelsky and A. A. Stepanov. Path-planning strategies for a point mobile automaton moving amidst unknown obstacles of arbitrary shape. *Algorithmica*, 2:403–430, 1987.

[Sch01] S. Schuierer. Lower bounds in on-line geometric searching. *Comput. Geom. Theory Appl.*, 18:37–53, 2001.

[Sha52] Claude E. Shannon. Presentation of a maze solving machine. In H. von Foerster, M. Mead, and H. L. Teuber, editors, *Cybernetics: Circular, Causal and Feedback Mechanisms in Biological and Social Systems, Transactions Eighth Conference, 1951*, pages 169–181, New York, 1952. Josiah Macy Jr. Foundation. Reprint in [Sha93].

[Sha93] Claude E. Shannon. Presentation of a maze solving machine. In Neil J. A. Sloane and Aaron D. Wyner, editors, *Claude Shannon: Collected Papers*, volume PC-03319. IEEE Press, 1993.

[SM92] A. Sankaranarayanan and I. Masuda. A new algorithm for robot curvefollowing amidst unknown obstacles, and a generalization of maze-searching. In *Proc. 1992 IEEE Internat. Conf. on Robotics and Automation*, pages 2487–2494, 1992.

[Sut69] Ivan E. Sutherland. A method for solving arbitrary wall mazes by computer. *IEEE Trans. on Computers*, 18(12):1092–1097, 1969.

[SV90a] A. Sankaranarayanan and M. Vidyasagar. A new path planning algorithm for a point object amidst unknown obstacles in a plane. In *Proc. 1990 IEEE Internat. Conf. on Robotics and Automation*, pages 1930–1936, 1990.

[SV90b] A. Sankaranarayanan and M. Vidyasagar. Path planning for moving a point object amidst unknown obstacles in a plane: A new algorithm and a general theory for algorithm developments. In *Proceedings of 1990 IEEE Conf. on Decision and Control*, pages 1111–1119, 1990.

[SV91] A. Sankaranarayanan and M. Vidyasagar. Path planning for moving a point object amidst unknown obstacles in a plane: The universal lower bound on the worst case path lengths and a classification of algorithms. In *Proc. 1991 IEEE Internat. Conf. on Robotics and Automation*, pages 1734–1741, 1991.

[Wal86] Wolfgang Walter. *Gewöhnliche Differentialgleichungen*. Springer, 1986.

[Web07] Maximilian Weber. Online suche auf beschränkten sternen. Diplomarbeit, Rheinische Friedrich-Wilhelms-Universität Bonn, 2007.

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