Problem Set 10

For an undirected graph $G = (V, E)$ with weights $w : E \to \mathbb{R}_{\geq 0}$ the Maximum-Cut Problem is the problem of finding a partition of $V$ into disjoint sets $V_0$ and $V_1$ that maximizes

$$w(V_0, V_1) := \sum_{\{u, v\} \in E, u \in V_0 \land v \in V_1} w(e)$$

over all possible partitions. We call a partition $(V_0, V_1)$ also a cut and we say that $w(V_0, V_1)$ is the weight of the cut $(V_0, V_1)$.

We consider the simple local search algorithm FLIP for the Maximum-Cut Problem that starts with an arbitrary cut $(V_0, V_1)$ and iteratively increases the weight of the cut by moving one vertex from $V_0$ to $V_1$ or vice versa, as long as such an improvement is possible. For $i \in \{0, 1\}$ and a vertex $v \in V_i$ the switch corresponding to $v$ is moving $v$ from $V_i$ to $V_1-i$, which creates a new cut $(V'_i, V'_{1-i}) = (V_i \setminus \{v\}, V_{1-i} \cup \{v\})$. A switch is improving if it increases the weight of the cut, i.e. $w(V'_i, V'_{1-i}) > w(V_0, V_1)$. The algorithm FLIP stops when the current cut does not admit an improving switch anymore.

**Problem 1**

Show that FLIP outputs a cut whose weight is at least half the weight of the maximum cut.

**Problem 2**

Show a pseudo-polynomial upper bound on the running time of FLIP for instances in which all weights are integers.

**Problem 3**

(a) Assume that the weights $w : E \to [0, 1]$ are $\phi$-perturbed numbers. Give an upper bound on the expected number of iterations of FLIP on instances in which $G$ has maximal degree $\delta$.

(b) For which values of $\delta$ is the expected number of iterations from part (a) polynomial.

**Problem 4**

Let $V$ be a set of $n$ points in $[0, 1] \times [0, 1]$. For every pair of points $u, v \in V$, let $d(u, v)$ be defined as the Euclidean distance between $u$ and $v$. Show that the length of the optimal TSP tour with respect to $d$ is $O(\sqrt{n})$. 