Problem Set 8

Problem 1
Let $n \in \mathbb{N}$, $m \in \{n, \ldots, n^2\}$, and $\phi \in \mathbb{N}$ be given. Find a minimum-cost flow network $G = (V, E, u, c)$ with $2n + 2$ vertices, $m + 2n$ edges, and $\phi$-perturbed edge costs on which the SSP algorithm needs $m$ iterations for every realization of the edge costs. You may assume that for any nodes $u$ and $v$ the lengths of all possible $u$-$v$ paths are pairwise distinct (Property 5.11). 

Hint: This can be achieved such that there is a direct correspondence between the iterations and the edges that are neither connected to the source nor the sink.

Problem 2
Let $\phi \geq 5$ be given.

(a) Find a minimum-cost flow network $G = (V, E, u, c)$ with 4 vertices, containing a single source $s$ and a single sink $t$, 5 edges, and $\phi$-perturbed edge costs such that for one of the edges $e \in E$ the SSP algorithm will augment via a path containing $e$ and via another path containing $\overrightarrow{e}$ for every realization of the edge costs.

(b) Let $\tilde{G}$ be the network created when the edge $e$ from part (a) of the exercise is replaced by some single-source-single-sink-sub-network $G' = (V', E', u', c')$. What properties does $G'$ need to have in order to guarantee that the SSP algorithm, when run on $\tilde{G}$, first augments via a series of paths that create a maximum flow in $G'$ and then via another series of paths after that no flow is left on any of the edges of $G'$?

Problem 3
Let $n \in \mathbb{N}$, $m \in \{n, \ldots, n^2\}$, and $\phi \in \mathbb{N}$ with $\phi \in O(2^n)$ be given. Start with the instance from Problem 1 and use the result from Problem 2 to create an iterative method to create instances that contain $O(n)$ vertices, $O(m)$ edges, and $\phi$-perturbed edge costs, on which the SSP algorithm augments via $\Omega(m \cdot \phi)$ many paths for every realization of the edge costs that satisfies Property 5.11.

How long can this iterative method be used? What are possible restricting parameters?

Problem 4
Prove that for given positive integers $n$, $m \in \{n, \ldots, n^2\}$, and $\phi \leq 2^n$ there exists a minimum-cost flow network with $O(n)$ nodes, $O(m)$ edges, and $\phi$-perturbed edge costs on which the SSP algorithm requires $\Omega(m \cdot \min\{n, \phi\} \cdot \phi)$ augmentation steps.