Problem 1
Let \( n \) points be placed uniformly at random on the boundary of a circle of circumference 1. These \( n \) points divide the circle into \( n \) arcs.

(a) What is the average arc length?

(b) Let \( x \) denote an arbitrary fixed point on the circle. What is the expected length of the arc that contains the point \( x \)?

Problem 2
Find an algorithm for the knapsack problem that runs in the worst case in time \( O(nP) \), where \( n \) is the number of items, all profits \( p_1, \ldots, p_n \in \mathbb{N} \) are natural numbers, and \( P := \sum_{i=1}^{n} p_i \). Why does the existence of such an algorithm not prove \( P = NP \)?

Problem 3
For an instance of the knapsack problem with profits \( p \in \mathbb{R}^n_{\geq 0} \), weights \( w \in \mathbb{R}^n_{\geq 0} \), and capacity \( W \in \mathbb{R} \), we define the winner gap \( \Delta \) to be the difference in profit between the best solution \( x^* \) and the second best solution \( x^{**} \). Formally, let \( \Delta := p^T x^* - p^T x^{**} \), where

\[
\begin{align*}
x^* := & \arg \max \{ p^T x \mid x \in \{0,1\}^n \text{ and } w^T x \leq W \} \\
x^{**} := & \arg \max \{ p^T x \mid x \in \{0,1\}^n \text{ and } w^T x \leq W \text{ and } x \neq x^* \}.
\end{align*}
\]

We assume that there are at least two feasible solutions. Then \( \Delta \) is well-defined. Let the weights be arbitrary and let the profits be \( \phi \)-perturbed numbers from \([0,1]\), i.e., each profit \( p_i \) is chosen independently according some probability density \( f_i : [0,1] \to [0,\phi] \) for some fixed \( \phi \geq 1 \). Show that for any \( \epsilon > 0 \)

\[
\Pr[\Delta \leq \epsilon] \leq n\phi \epsilon.
\]

Problem 4
Give an implementation of the Nemhauser-Ullmann algorithm in Java or C++ with running time \( O(\sum_{i=0}^{n-1} |\mathcal{P}_i|) \), where \( n \) denotes the number of items and \( \mathcal{P}_i \) denotes the Pareto set of the restricted instance that consists only of the first \( i \) items.

Use your implementation to generate the Pareto set of instances in which all profits and weights are chosen uniformly at random from \([0,1]\) for \( n = 10, 20, 30, \ldots \). How does the number of Pareto-optimal solutions and the running time depend on \( n \) in your experiments.