Exercise 22: Voronoi edges of $k^{th}$-order Voronoi diagrams (4 points)
Consider a Voronoi edge $e$ between two adjacent Voronoi regions $\text{VR}_k(H_1, S)$ and $\text{VR}_k(H_2, S)$, where $S$ is a set of $n$ point sites in the Euclidean plane. Please prove the following.

1. $|H_1 \setminus H_2| = |H_2 \setminus H_1| = 1$

2. The circle centered at a point $x$ in $e$ and touching $p$ and $q$, where $H_1 \setminus H_2 = \{p\}$ and $H_2 \setminus H_1 = \{q\}$, encloses exactly $k - 1$ sites of $S$.

(Hint: Consider $\text{VR}_{k-1}(H, S)$ and $V_i(S \setminus H)$, where $e \cap \text{VR}_{k-1}(H, S) \neq \emptyset$.)

Exercise 23: Numbers of vertices, edges, and faces of $V_k(S)$ (12 points)
Let $S$ be a set of $n$ point sites in the Euclidean plane satisfying a general position assumption that no three sites are on the same line and no four sites are on the same circle. For $1 \leq i \leq n - 1$, let $N_i, E_i, I_i, B_k, S_i$ be the numbers of faces, edges, vertices, bounded regions, and unbounded faces of $V_i(S)$, respectively, and let $S_0$ be 0. Please prove the following:
1. \( E_k = 3(N_k - 1) - S_k \) and \( I_k = 2(N_k - 1) - S_k \). (Hint: Euler formula. Due the general position assumption, the degree of a Voronoi vertex is 3).

2. \( N_1 = n \), and \( N_2 = 3(n - 1) - S_1 \), and \( N_k = 3(N_{k-1} - 1) - S_{k-1} - 2 \sum_{i=1}^{k-2} (-1)^{k-2-i} (2(N_i - 1) - S_i) \) implies

\[
N_k = 2k(n - k) + k^2 - n + 1 - \sum_{i=0}^{k-1} S_i.
\]

(Hint: By induction on \( k \))

3. \( \sum_{k=1}^{n-1} B_k = \binom{n-1}{3} \) (Hint: \( \sum_{k=1}^{n-1} I_k = 2\binom{n}{3} \) and \( \sum_{k=1}^{n-1} S_k = 2\binom{n}{2} \))

4. Let \( I'_k \) be the number of new vertices of \( V_k(S) \). Prove that \( I'_k = 2k(n - k) + k^2 - k - \sum_{i=1}^{k} S_i \). (Hint: \( N_{k+2} = E_{k+1} - 2I'_k. \))

Exercise 24: Relation between \( V_i(S) \) and \( V_{i+1}(S) \) (4 points)

Assume \( VR_i(H, S) \) has \( m \) adjacent regions \( VR_i(H_j, S) \), \( 1 \leq j \leq m \). Let \( Q \) be \( \bigcup_{1 \leq j \leq m} H_j \setminus H \). Prove that \( V_{i+1}(S) \cap VR_i(H, S) = V_1(Q) \cap VR_i(H, S) \).

(Hint: prove that for all site \( r \in (S \setminus H) \setminus Q \), \( VR_1(r, S \setminus H) \cap VR_k(H, S) = \emptyset \).

You can first assume the contrary that \( \exists r \in (S \setminus H) \setminus Q \ VR_1(r, S \setminus H) \cap VR_k(H, S) \neq \emptyset \), and then show that it will lead to a contradiction. For any point \( x \in VR_1(r, S \setminus H) \cap VR_k(H, S) \), \( r \) will intersect a Voronoi edge \( e \) between \( VR_i(H, S) \) and \( VR_i(H_j, S) \) for some \( j \in \{1, \ldots, m\} \). Let \( y \) be the intersection point between \( r \) and \( e \). Discuss nearest neighbors of \( y \), which will lead to a contradiction from the viewpoint of \( e \) and the viewpoint of \( VR_1(r, S \setminus H) \).)

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