$k^{\text{th}}$-order Voronoi Diagrams

References:


Given a set $S$ of $n$ point sites in the Euclidean plane, the $k^{\text{th}}$-order Voronoi diagram $V_k(S)$ is a planar subdivision such that

- each region is associated with a $k$-element subset $H$ of $S$ and denoted by $VR_k(H, S)$.
- all points in $VR_k(H, S)$ share the same $k$ nearest sites $H$ among $S$.

$$V_2(S)$$

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(p, s) {p, q}

q

{q, s}

r

{r, s} {q, r}
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Property 1
Consider a Voronoi edge $e$ between $\text{VR}_k(H_1, S)$ and $\text{VR}_k(H_2, S)$. $H_1$ and $H_2$ only differ by one site. Let $H_1 \setminus H_2$ be $\{p\}$ and $H_2 \setminus H_1$ be $\{q\}$. For all points $x \in e$, $H_1 \cap H_2$ are the $k - 1$ nearest sites of $x$ and both $p$ and $q$ are the $k$th nearest sites of $x$.

![Diagram]

General Position Assumption
- no more than than sites are on the same line → $V_k(S)$ is connected.
- no more than three sites are on the same circle → the degree of a Voronoi vertex is exactly 3.

Definition 1
Consider a Voronoi vertex $v$ among $\text{VR}_k(H_1, S)$, $\text{VR}_k(H_2, S)$, and $\text{VR}_k(H_3, S)$.
- $v$ is new if $|H_1 \cup H_2 \cup H_3| = k + 2$. $H_1 = H \cup \{p\}$, $H_2 = H \cup \{q\}$, $H_3 = H \cup \{r\}$, where $|H| = k - 1$. → the circle centered at $v$ and touching $p$, $q$, and $r$ will exactly enclose the $k - 1$ sites of $H$.
- $v$ is old if $|H_1 \cup H_2 \cup H_3| = k + 1$. $H_1 = H \cup \{p, q\}$, $H_2 = H \cup \{q, r\}$, $H_3 = H \cup \{p, r\}$, where $|H| = k - 2$. → the circle centered at $v$ and touching $p$, $q$, and $r$ will exactly enclose the $k - 2$ sites of $H$. 
Example

\[ H_1 = H \cup \{p\} \]
\[ H_2 = H \cup \{q\} \]
\[ H_3 = H \cup \{r\} \]
\[ |H| = 3 \]

\[ H_1 = H \cup \{p, q\} \]
\[ H_2 = H \cup \{q, r\} \]
\[ H_3 = H \cup \{p, r\} \]
\[ |H| = 2 \]

Property 2

\( v \) is a Voronoi vertex among \( \text{VR}_k(H_1, S) \), \( \text{VR}_k(H_2, S) \), and \( \text{VR}_k(H_3, S) \)

(a) \( v \) is new

\[ \rightarrow v \text{ is an old Voronoi vertex among } \text{VR}_k(H_1 \cup H_2, S), \text{VR}_k(H_2 \cup H_3, S), \text{VR}_k(H_3 \cup H_1, S). \]

(b) \( v \) is old

\[ \rightarrow v \text{ belongs to } \text{VR}_k(H_1 \cup H_2 \cup H_3). \]
Property 3
Consider an edge $e$ between $\text{VR}_k(H_1, S)$ and $\text{VR}_k(H_2, S)$. Then all points $x \in e$ belong to $\text{VR}_k(H_1 \cup H_2)$.

Sketch of proof:
Let $H_1 \setminus H_2$ be $\{p\}$ and $H_2 \setminus H_1$ be $\{q\}$. Since $e$ is a part of the bisector $B(p, q)$ between $p$ and $q$, the circle centered at $x$ and touching $p$ and $q$ will enclose all the $k-1$ sites of $H_1 \cap H_2$. Therefore, $(H_1 \cap H_2) \cup \{p, q\} = H_1 \cup H_2$ are exactly the $k+1$ nearest sites of $x$.

Definition 2
For a Voronoi edge $e$ of $V_k(S)$, if one endpoint of $e$ is an old Voronoi vertex, $e$ is called old; otherwise, $e$ is called new.

Property 4
New vertices of $V_k(S)$ decompose $V_k(S)$ into two kinds of connected components:

1. a new Voronoi edge
2. a connected subgraph whose internal nodes are old Voronoi vertices

Each kind induces a Voronoi region of $V_{k+1}(S)$. (The former comes from Property 2 (a) and Property 3, and the latter comes from Property 2(b) and Property 3.)

Definition 3
For $i > 1$, Voronoi regions $\text{VR}_i(H, S)$ of $V_i(S)$ can be categorized into two types:

- **type-1**: $\text{VR}_i(H, S)$ contains one new edge of $V_{i-1}(S)$.
- **type-2**: $\text{VR}_i(H, S)$ contains old vertices of $V_{i-1}(S)$. 
Example

Type-1

VR₂({q, s}, S) is a type-1 region because it contains one new edge of V₁(S)

Type-2

Both VR₃({p, q, s}, S) and VR₃({q, r, s}, S) are type-2 regions because they contain old vertices of V₂(S)
Lemma 1
For \( i > 1 \), \( V_{i-1}(S) \cap VR_i(H, S) \) is a tree. \( V_{i-1}(S) \cap VR_i(H, S) = V_{i-1}(H) \cap VR_i(H, S) \)

**Sketch of proof**
- all points in \( VR_i(H, S) \) share the same \( i \) nearest sites.
- \( V_{i-1}(S) \) partitions \( VR_i(H, S) \) into at most \( t \) sub-regions, and \( t < i \).
- For \( 1 \leq j \leq t \), let \( R_j \) be a sub-region of \( V_{i-1}(S) \cap VR_i(H, S) \), let \( H_j \) be the \((i-1)\)-element subset of \( S \) such that \( R_j = VR_{i-1}(H_j, S) \cap VR_i(H, S) \), and let \( H \setminus H_j \) be \( \{s_j\} \).
- For all points \( x \) in \( R_j \), \( H_j \) are the \((i-1)\) nearest sites of \( x \), and \( s_j \) is the \( i \)th nearest site of \( x \).
- In other words, \( s_j \) is the farthest site of \( x \) among \( H \).
- \( V_{i-1}(S) \) forms the farthest site Voronoi diagram of \( H \) inside \( VR_i(H, S) \), i.e., \( V_{i-1}(S) \cap VR_i(H, S) = V_{i-1}(H) \cap VR_i(H, S) \).
- The farthest-site Voronoi diagram is a tree.
- By Property 4, \( V_{i-1}(S) \cap VR_i(H, S) \) is a connected component, and thus \( V_{i-1}(H) \cap VR_i(H, S) \) is a tree.

Corollary 1
If \( VR_i(H, S) \) contains \( m \) old Voronoi vertices of \( V_{i-1}(S) \), \( VR_i(H, S) \) contains \( 2m + 1 \) old Voronoi edges of \( V_{i-1}(S) \).

**Sketch of proof**
- By the generation position assumption, the degree of a Voronoi vertex is 3.
- By Lemma 1, \( V_{i-1}(S) \cap VR_i(H, S) \) is a tree.

Euler formular for a planar subdivision
\[
\nu - \varepsilon + \varphi = 1 + c,
\]
where \( \nu \) is \# of vertices, \( \varepsilon \) is \# of edges, \( \varphi \) is \# of faces, and \( c \) is \# of connected component.
Corollary 2
Under the general position assumption,

- \( E_k = 3(N_k - 1) - S_k \)
- \( I_k = 2(N_k - 1) - S_k \),

where \( E_k \) is \# of edges, \( I_k \) is \# of vertices, \( N_k \) is \# of faces, and \( S_k \) is \# of unbounded faces of \( V_k(S) \).

Theorem 1
Given a set \( S \) of \( n \) point sites in the Euclidean plane, the total number \( N_k \) of regions in \( V_k(S) \) is \( 2k(n - k) + k^2 - n + 1 - \sum_{i=1}^{k-1} S_i \), where \( S_i \) is \# of unbounded regions in \( V_i(S) \), and \( S_0 \) is defined to be 0.

proof

- \( I_i, I'_i \) and \( I''_i \) are \# of vertices, new vertices, and old vertices of \( V_i(S) \), respectively.
- \( E_i, E'_i \) and \( E''_i \) are \# of edges, new edges, and old edges of \( V_i(S) \), respectively.
- \( N_i, N'_i \) and \( N''_i \) are \# of regions, type-1 regions, and type-2 regions of \( V_i(S) \), respectively.
- Since an old vertex of \( V_{i+1}(S) \) is a new vertex of \( V_i(S) \),
  \[
  I_{i+1} = I'_{i+1} + I''_{i+1} = I'_{i+1} + I'_i \\
  \rightarrow I''_{i+1} = I_{i+1} - I'_{i+1}
  \]
- \( I_1 = I'_1, E_1 = E'_1, \) and \( E_{i+1} = E'_{i+1} + E''_{i+1} \)
- Order \( N''_{i+2} \) type-2 regions of \( V_{i+2}(S) \), let \( m_j \) be the number of old vertices of \( V_{i+1}(S) \) inside the \( j \)th type-2 region of \( V_{i+2}(S) \), and let \( e_j \) be the number of edges of \( V_{i+1}(S) \) inside the \( j \)th type-2 region of \( V_{i+2}(S) \).
- \( \sum_{j=1}^{N''_{i+2}} m_j = I''_{i+1} = I'_i \) and \( \sum_{j=1}^{N''_{i+2}} e_j = E''_{i+1} \)
- By Corollay 1,
  \[
  E''_{i+1} = \sum_{j=1}^{N''_{i+2}} e_j = \sum_{j=1}^{N''_{i+2}} (2m_j + 1) = 2I'_i + N''_{i+2} \Rightarrow N''_{i+2} = E''_{i+1} - 2I'_i
  \]
• \( N_{i+2} = N_{i+2}' + N_{i+2}'' = E_{i+1}' + (E_{i+1}'' - 2I_i') = E_{i+1} - 2I_i' \)

• \( N_1 = n \) and \( N_2 = E_1' = E_1 = 3(n - 1) - S_1. \)

• Since \( N_{i+2} = E_{i+1}' - 2I_i', \ E_i = 3(N_i - 1) - S_i, \) and \( I_i = 2(N_i - 1) - S_i, \)
  \[ N_{k+2} = E_{k+1}' - 2I_k' = 3(N_{k+1} - 1) - S_{k+1} - 2I_k' \]
  \[ = 3(N_{k+1} - 1) - S_{k+1} - 2 \sum_{i=1}^{k} (-1)^{k-i} I_i \]
  \[ = 3(N_{k+1} - 1) - S_{k+1} - 2 \sum_{i=1}^{k} (-1)^{k-i} (2(N_i - 1) - S_i) \]

• By induction on \( k, \)
  \[ N_k = 2k(n - k) + k^2 - n + 1 - \sum_{i=1}^{k-1} S_i \]

**Theorem 2**

\[ N_k = O(k(n - k)) \]

• If \( k \leq n/2, \) by Theorem 1, \( N_k \) is trivially \( O(k(n - k)) \).

• If \( k > n/2, \) \( N_k \) depends on \( \sum_{i=1}^{k-1} S_i \)

• Since \( \sum_{i=1}^{n-1} S_i = n(n - 1), \ \sum_{i=1}^{k-1} S_i = n(n - 1) - \sum_{i=k}^{n-1} S_i \)

• Since \( S_i = S_{n-i}, \ \sum_{i=k}^{n-1} S_i = \sum_{i=1}^{n-k} S_i \)

• \( N_k = 2k(n - k) + k^2 - n + 1 - \sum_{i=1}^{k-1} S_i \)
  \[ = 2k(n - k) + k^2 - n + 1 - n(n - 1) + \sum_{i=k}^{n-1} S_i \]
  \[ = N_k = 2k(n - k) + k^2 - n + 1 - n(n - 1) + \sum_{i=1}^{n-k} S_i \]

• Since \( \sum_{i=1}^{n-k} S_i \leq (n - k)n \) (recall \# of \( \leq k \)-set),
  \[ N_k \leq 2k(n - k) + k^2 - n + 1 - n(n - 1) + (n - k)n = k(n - k) + 1 \]
Theorem 3
$V_{i+1}(S)$ can be obtained from $V_i(S)$ by taking $\text{VR}_i(H, S) \cap V_1(S \setminus H)$ for all $H \subseteq S$ such that $V_i(H, S)$ is non-empty.

Sketch of proof

- $V_1(S \setminus H) \cap \text{VR}_i(H, S) = V_{i+1}(S) \cap \text{VR}_i(H, S)$
  - all points in $\text{VR}_i(H, S)$ share the same $i$ nearest sites $H$ among $S$
  - all points in $\text{VR}_1(p, S \setminus H)$ share the same nearest site $p$ among $S \setminus H$.
  - all points in $\text{VR}_1(p, S \setminus H) \cap \text{VR}_i(H, S)$ share the same $i$ nearest sites $H$ and $(i+1)^{\text{th}}$ nearest site $p$ among $S$, implying that $\text{VR}_1(p, S \setminus H) \cap \text{VR}_i(H, S) \subseteq \text{VR}_{i+1}(H \cup \{p\}, S)$
  - It is trivial that $\text{VR}_{i+1}(H \cup \{p\}, S) \cap \text{VR}_i(H, S) \subseteq \text{VR}_1(p, S \setminus H)$,
  - $\text{VR}_1(p, S \setminus H) \cap \text{VR}_i(H, S) = \text{VR}_{i+1}(H \cup \{p\}, S) \cap \text{VR}_i(H, S)$ for $\forall p \in H$

Corollary 3
Assume $\text{VR}_i(H, S)$ has $m$ adjacent regions $\text{VR}_i(H_j, S)$, $1 \leq j \leq m$. Let $Q = \bigcup_{1 \leq j \leq m} H_j \setminus H$. Then $V_{i+1}(S) \cap \text{VR}_i(H, S) = V_1(Q) \cap \text{VR}_i(H, S)$

The proof will be an exercise.

Compute $V_{i+1}(S)$ from $V_i(S)$

- For each nonempty region $\text{VR}_i(H, S)$, compute $V_1(Q) \cap \text{VR}_i(H, S)$ where $\text{VR}_i(H, S)$ has $m$ adjacent regions $\text{VR}_i(H_j, S)$, $1 \leq j \leq m$, and $Q$ is $\bigcup_{1 \leq j \leq m} H_j \setminus H$. 
Lemma 2
$V_{i+1}(S)$ can be obtained from $V_i(S)$ in $O(i(n - i) \log n)$ time.

**Sketch of proof**

- $V_1(Q)$ can be computed in $|Q| \log |Q|$ time.
- $|Q| \leq |\partial VR_i(H, S)|$ where $\partial VR_i(H, S)$ is the boundary of $VR_i(H, S)$
- $O(|\partial VR_i(H, S)| \log |\partial VR_i(H, S)|)$

$$= \log n \sum_{H \subset S, |H| = i, VR_i(H, S) \neq \emptyset} O(|\partial VR_i(H, S)|)$$

Theorem 4
$V_k(S)$ can be computed in $O(k^2 n \log n)$ time.

**Sketch of proof**

- $V_1(S)$ can be computed in $O(n \log n)$
- $O(n \log n) + \sum_{i=1}^{k-1} O(i(n - i) \log i) = O(k^2 n \log n)$. 
Construction by Geometric Duality and Arrangement

**Definition 4 (Bisectors)**
- For two sites, \( p, q \in S \), the bisector \( B(p, q) \) is \( \{x \in \mathbb{R}^2 \mid d(x, p) = d(x, q)\} \).
- For a site \( p \in S \), let \( B_p \) be \( \{B(p, q) \mid q \in S \setminus \{p\}\} \).

**Definition 5**
For a site \( p \in S \), the \( k \)-neighborhood of \( p \) is \( \bigcup_{p \in H, H \subseteq S, |H| = k} \text{VR}_k(H, S) \) and denoted by \( \text{VN}_k(p, S) \).

**Property 5**
\[ V_k(S) = \bigcup_{p \in S} \partial \text{VN}_k(p, S) \]

**Lemma 3**
\( \text{VN}_k(p, S) \) is connected and each edge of \( \partial \text{VN}_k(p, S) \) is a part of the bisector \( B(p, q) \) for some \( q \in S \setminus \{p\} \).
The proof could be a bonus task.

**Lemma 4**
Consider an edge of \( \partial \text{VN}_k(p, S) \). For any point \( x \in e \), \( \overline{px} \) intersects exactly \( k - 1 \) bisectors of \( B_p \).

*Sketch of proof*
- W.l.o.g, let \( e \) belong to \( \text{VR}_k(H_1, S) \cap \text{VR}_k(H_2, S) \) and let \( p \) belong to \( H_1 \setminus H_2 \).
- It is clear that \( H_1 \setminus \{p\} \) are the \( k - 1 \) nearest sites of \( x \).
- For any \( q \in H_1 \setminus \{p\} \), \( x \) belongs to \( D(q, p) \), i.e., \( \overline{px} \) intersects \( B(p, q) \). For any \( q \in S \setminus H_1 \), \( x \) does not belongs to \( D(q, p) \), i.e., \( \overline{px} \) does not intersects \( B(p, q) \).
Lemma 5
\[ \partial VN_k(p, S) = SK_k(p, B_p) \]

Therefore, computing \( V_k(S) \) is equivalent to computing \( SK_k(p, B_p) \) for all sites \( p \in S \).
Hereafter, we translate \( S \) such that \( p \) is located at \((0, 0)\), and let \( L \) be \( B_p \).
If we know all the vertices of \( SK_k(p, L) \) and their order along \( SK_k(p, L) \) (clockwise or counterclockwise, we can compute \( SK_k(p, L) \)).

Lemma 6
Under the general position assumption, for a vertex \( v \) of \( SK_k(p, B_p) \), \( pv \) intersects \( k - 1 \) or \( k - 2 \) lines of \( B_p \).

Geometric Duality
Consider a function \( \Psi \). For a point \( x = (a, b) \) except the origin, \( \Psi(x) \) is a line \( : ax_1 + bx_2 = 1 \), and for a line \( l : ax_1 + bx_2 = 1 \), \( \Psi(x) \) is a point \((a, b)\).

Lemma 7
• For an edge \( e \) of \( SK_k(p, B_p) \) and any point \( x \in e \), \( \Psi(x) \) partitions the plane such that one half-plane contains the origin and exactly \( k - 1 \) points of \( \Psi(B_p) \).
• For a vertex \( v \) of \( SK_k(p, B_p) \), \( \Psi(v) \) partitions the plane such that one half-plane contains the origin and \( k - 1 \) or \( k - 2 \) points of \( \Psi(B_p) \).
Example
For \( q \in S \setminus \{ p \} \), let \( p_q \) be \( \Psi(B(p, q)) \). Consider \( n = 8 \) and \( k = 4 \).

\[
l_{q,r} \text{ corresponds to a new Voronoi vertex among } VR_k(H_1, S), VR_k(H_2, S), \text{ and } VR_k(H_3, S), \text{ where } H_1 = H \cup \{ p \}, H_2 = H \cup \{ q \}, H_3 = H \cup \{ r \}, \text{ and } H = \{ s, t, u \}.
\]

\[
l \text{ corresponds to a point on a Voronoi edge between } VR_k(H_1, S) \text{ and } VR_k(H_2, S), \text{ where } H_1 = H \cup \{ p \}, H_2 = H \cup \{ q \}, \text{ and } H = \{ s, t, u \}.
\]

\[
l_{q,s} \text{ corresponds to an old Voronoi vertex among } VR_k(H'_1, S), VR_k(H'_2, S), \text{ and } VR_k(H'_3, S), \text{ where } H'_1 = H' \cup \{ p, s \}, H'_2 = H' \cup \{ q, s \}, H'_3 = H \cup \{ p, q \}, \text{ and } H' = \{ t, u \}. \text{ (Note } H'_1 = H_1 \text{ and } H'_2 = H_2 \text{.)}
\]
Let $v_1, v_2, \ldots$ be a sequence of vertices of $\text{SK}_k(p, B_p)$ along the counterclockwise order.

We consider how to compute $v_{i+1}$ from $v_i$.

- W.l.o.g., we let $v_i$ be the intersection between $B(p, q)$ and $B(p, r)$ and $v_{i+1}$ be $B(p, q)$ and $B(p, s)$. But we do not know $s$.
- Similarly, for each $q \in S \setminus \{p\}$, let $p_q$ be $\Psi(B(q, p))$.
- $\Psi(v_i)$ is a straight line passing through $p_q$ and $p_r$.
- Let $l$ be $\Psi(v_i)$, and rotate $l$ at $p_q$ in the direction such that one half-plane contains the origin and exactly $k - 1$ points of $\Psi(B_p)$.
- The rotation will hit $p_s$ first and we obtain $v_{i+1}$.
- During the rotation, $l$ partition $\Psi(B_p \setminus \{B(p, q)\})$ into the same 2 sets.

**Property 6**

Let $e$ be an edge of $\text{SK}_k(p, S)$ and belong to $B(p, q)$. Let $v$ be an endpoint of $e$ and $v$ be an intersection between $B(p, q)$ and $B(p, s)$. For any point $x \in e$, let $P_1$ and $P_2$ be the 2-partition of $\Psi(B_p \setminus \{B(p, q)\})$ formed by $\Psi(x)$. Then, $\Psi(B(p, s))$ must be one of four tangent points between $\Psi(B(p, q))$ and the two convex hulls of $P_1$ and $P_2$. 
Lemma 8
$SK_k(p, B_p)$ can be constructed in $O(n \log n + |SK_k(p, B_p)| \log n)$ time.

Sketch of proof

- After the sorting, it takes $O(n)$ time to compute a vertex of $SK_k(p, B_p)$ and then view the vertex as the begining vertex $v_1$.
- It sufficient to analyze the time for computing $v_{i+1}$ from $v_i$.
- Assume that $v_i$ is an intersection between $B(p, q)$ and $B(p, r)$.
- Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the 2-partion of $\Psi(B_p \setminus \{p\})$ formed by $\Psi(v_i)$ and let $\mathcal{P}_1$ belong to the half-plane containing the origin.
- If $v_i$ is a new Voronoi vertex, $|\mathcal{P}_1| = k - 1$.
  - let $l$ be $\Psi(v_i)$
  - rotate $l$ at $\Psi(B(p, q))$ such that $\mathcal{P}_1$ and $\Psi(B(p, r))$ belongs to different half-planes formed by $l$.
  - Determine that $l$ first touches the convex hull of $\mathcal{P}_1$ or that of $\mathcal{P}_2 \cup \{\Psi(B(p, r))\}$
  - Let $\Psi(B(p, s))$ be the first touched point of the first touched convex hull. Then $v_{i+1}$ is the intersection between $B(p, q)$ and $B(p, s)$.
- Otherwise, $v_i$ is an old Voronoi vertex, and $|\mathcal{P}_1| = k - 2$.
  - let $l$ be $\Psi(v_i)$
  - rotate $l$ at $\Psi(B(p, q))$ such that $\mathcal{P}_1$ and $\Psi(B(p, r))$ belong to the same half-plane formed by $l$.
  - Determine that $l$ first touches the convex hull of $\mathcal{P}_1 \cup \{\Psi(B(p, r))\}$ or that of $\mathcal{P}_2$
  - Let $\Psi(B(p, s))$ be the first touched point of the first touched convex hull. Then $v_{i+1}$ is the intersection between $B(p, q)$ and $B(p, s)$.
- Brodal and Jacob proposed a dynamic structure for the convex hulls allowing insertion, deletion, and tangent query in amorted $O(\log n)$ time.
- It takes $O(n \log n)$ time to compute the two initial convex hulls.
- There are $O(|SK_k(p, B_p)|$ insertions, deletions, and tangent queries.
Theorem 5

$V_k(S)$ can be computed in $O(n^2 \log n + k(n - k) \log n)$ time.

Sketch of proof

- $V_k(S) = \bigcup_{p \in S} SK_k(p, B_p)$.
- $\sum_{p \in S} O(n \log n + |SK_k(p, B_p)| \log n) = O(n^2 \log n + k(n - k) \log n)$