6. Construction of AVD

Finite Part of AVD

- Let Γ be a simple closed curve such that all intersections between bisecting curve lie inside the inner domain of Γ
- Consider a site ∞, define \( J(p, ∞) = J(∞, p) \) to be Γ for all sites \( p \in S \), and \( D(∞, p) \) to be the outer domain of Γ for all sites \( p \in S \).

Incremental Construction

- Let \( s_1, s_2, \ldots, s_n \) be a random sequence of \( S \)
- Let \( R_i = \{∞, s_1, s_2, \ldots, s_i\} \)
- Iteratively construct \( V(R_2), V(R_3), \ldots, V(R_n) \)

General Position Assumption

- No \( J(p, q), J(p, r) \) and \( J(p, t) \) intersect the same point for any four distinct sites, \( p, q, r, t \in S \)
  \( \Rightarrow \) Degree of a Voronoi vertex is 3

Remark

- For \( 1 \leq i \leq n \) and for all sites \( p \in R_i \), \( VR(p, R_i) \) is simply connected, i.e., path connected and no hole
- If \( J(p, q) \) and \( J(p, r) \) intersect at a point \( x \), \( J(q, r) \) must pass through \( x \)
Basic Operations

- Given $J(p, q)$ and a point $v$, determine $v \in D(p, q)$, $v \in J(p, q)$, or $v \in D(q, p)$
- Given a point $v$ in common to three bisecting curves, determine the clockwise order of the curves around $v$
- Given points $u \in J(p, q)$ and $w \in J(p, r)$ and orientation of these curves, determine the first point of $J(p, r) |_{(w, \infty]}$ crossed by $J(p, q)|_{(v, \infty]}$
- Given $J(p, q)$ with an orientation and points $v, w, x$ on $J(p, q)$, determine if $v$ come before $w$ on $J(p, q) |_{(x, \infty]}$

Notation: Give a connected subset $A$ of $R^2$, int$A$, bd$A$, and cl$A$ mean the interior, the boundary, and the closure of $A$, respectively.

Conflict Graph $G(R)$, where $R$ is $R_i$ for $2 \leq i \leq n$

- bipartitle graph $(U, V, E)$
- $U$: Voronoi edges of $V(R)$
- $V$: Sites in $S \setminus R$
- $E : \{(e, s) \mid e \in V(R), s \in S \setminus R, e \cap VR(s, R \cup \{s\}) \neq \emptyset\}$
  
  - a conflict relation between $e$ and $s$.

Remark:

a Voronoi edge is defined by 4 sites under the general position assumption
Lemma 1
Let $R \subseteq S$ and $t \in S \setminus R$. Let $e$ be the Voronoi edge between $VR(p, R)$ and $VR(q, R)$. $e \cap VR(t, R \cup \{t\}) = e \cap R(t, \{p, q, r\})$. (Local Test is enough)

**Proof:**

$\subseteq$: Immediately from $VR(t, R \cup \{t\}) \subseteq VR(t, \{p, q, t\})$

$\supseteq$: Let $x \in e \cap VR(t, \{p, q, t\})$

- Since $x \in e$, $x \in VR(p, R) \cup VR(q, R)$ and $x \notin VR(r, R) \supseteq VR(r, R \cup \{t\})$ for any $r \in R \setminus \{p, q\}$.
- Since $x \in VR(t, \{p, q, t\})$, $x \notin VR(p, \{p, q, t\}) \cup VR(q, \{p, q, t\}) \supseteq VR(p, R \cup \{t\}) \cup VR(q, R \cup \{t\})$
- $x \notin VR(r, R \cup \{t\})$ for any site $r \in R \rightarrow x \in VR(t, R \cup \{t\})$

Insertiong $s \in S \setminus R$ to compute $V(R \cup \{s\})$ and $G(R \cup \{s\})$ from $V(R)$ and $G(R)$. Handle a conflict between $s$ and a Voronoi edge $e$ of $VR(R)$

Lemma 2
$cl e \cap cl VR(s, R \cup \{s\}) \neq \emptyset$ implies $e \cap VR(s, R \cup \{s\}) = \emptyset$

**proof**

- Let $x$ belong to $cl e \cap cl VR(s, R \cup \{s\})$
- $x$ is an endpoint of $e$:
  - $x$ is the intersection among three curves in $R$
  - For any $r \in R$, $J(s, r)$ cannot pass through $x$ due to the general position assumption
  - $x \in D(s, r) \rightarrow$ the neighborhood of $x \in D(s, r)$
  - $\exists y \in e$ belongs to $VR(s, R \cup \{s\})$
- $x \in e \cap bd VR(s, R \cup \{s\})$
  - $x \in J(p, q) \cap J(s, r)$
  - a point $y \in e$ in the neighborhood of $x$ such that $y \in VR(s, R \cup \{s\})$
Let $Q$ be $\text{VR}(s, R \cup \{s\})$

**Lemma 3**

$Q = \emptyset$ if and only if $\text{deg}_{G(R)}(s) = 0$

**Proof** ($\rightarrow$) If $Q = \emptyset$, $\text{deg}_{G(R)}(s) = 0$

($\leftarrow$)

- $\text{deg}_{G(R)}(s) = 0$ implies $\text{cl } Q \subseteq \text{int } \text{VR}(r, R)$ for some $r \in R$
- $\text{VR}(r, R \cup \{s\}) = \text{VR}(r, R) - Q$
- Since $\text{VR}(r, R \cup \{s\})$ must be simply connected, $Q = \emptyset$

**Lemma 4**

Let $I$ be $V(R) \cap \text{bd } Q$.

$I$ is a connected set which intersects $\text{bd } Q$ in at least two points.

**Proof:**

- $\text{bd } Q$ is a closed curve which does not go through any vertex of $V(R)$ due to the general position assumption.
- Let $I_1, I_2, \ldots, I_k$ be connected components of $I$
- Claim: $I_j$, $1 \leq j \leq k$, contains two points of $\text{bd } Q$.
  - If $I_j$ contains no point, $I_j \subseteq \text{int } Q$. In other words, for some $r \in R$, $\text{VR}(r, R)$ contains $I_j$, contradicting that $\text{VR}(r, R)$ must be simply connected.
  - If $I_j$ intersects exactly one point $x$ on $\text{bd } Q$, let $e$ be the Voronoi edge of $V(R)$ which contains $x$. Then both sides of $e$ belong to the same Voronoi region. There exists a contradiction.
• Assume the contrary that \( k \geq 2 \)
  
  - There is a path \( P \subseteq \text{cl } Q - (\cup_{1 \leq j \leq k} I_j) \) connects two points on \( \text{bd } Q \) such that one component of \( Q - P \) contains \( I_1 \) and the other component contains \( I_2 \).
  
  - Let \( x, y \) be the two endpoints of \( P \) and let \( r \in R \) such that \( P \subseteq \text{VR}(r, R) \).
  
  - Since \( x, y \notin V(R) \), \( \text{VR}(r, R) - Q \neq \emptyset \rightarrow x, y \in \text{cl } \text{VR}(r, R) \).
  
  - Since \( x, y \in \text{cl } \text{VR}(r, R) \), there is a path \( P' \subseteq \text{VR}(r, R) \) with endpoints \( x \) and \( y \).
  
  - \( P \circ P' \) is contained in \( \text{cl } \text{VR}(r, R) \) and contains either \( I_1 \) and \( I_2 \), contradicting \( \text{cl } \text{VR}(r, R) \) is simply connected.
Lemma 5
Let $e$ be an edge of $V(R)$. If $e \cap Q \neq \emptyset$,
- either $e \cap Q = V(R) \cap Q$ and $e \cap Q$ is a single component,
- or $e - Q$ is a single component

Proof
- Assume first $e \cap Q = V(R) \cap Q$
  - Since $V(R) \cap Q$ is connected, $e \cap Q$ is connected
- Assume next $e \cap Q \neq V(R) \cap Q$
  - At least one endpoint of $e$ is contained in $Q$
  - For every point $x \in e \cap Q$, one of the subpaths of $e$ connecting $x$ to an endpoint of $e$ must be contained in $Q$
  - $e - Q$ is a single component

Rough Idea
- Let $L$ be $\{e \in V(R) \mid (e, s) \in G(R)\}$
- For every edge $e \in L$, let $e'$ be $e - Q = e - V_R(s, R \cup \{s\})$. If $e$ is an edge between $V_R(p, R)$ and $V_R(q, R)$, $e' = e - D(s, p) = e - D(s, q)$
- Let $B$ be $\{x \in x$ is an endpoint of $e'$ but is not an endpoint of $e\} = V(R) \cap bd\ Q$
- $bd\ Q$ is a cyclic ordering on the points in $B$
**Step 1:** Compute $e'$ for each edge $e \in L$

**Step 2:** Compute $B$ and cyclic ordering on $B$ induced by $\text{bd } Q$

**Step 3:** Let $x_1, \ldots, x_k$ be the set $B$ in its cyclic ordering ($x_{k+1} = x_1$), and let $r_i$ such that $(x_i, x_{i+1}) \in \text{VR}(r_i, r)$

- For $1 \leq i \leq k$, add the part of $J(r_i, s)$ with endpoints $x_i$ and $x_{i+1}$

**Lemma 6**

$V(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in time $O(\deg_{G(R)}(s)+1)$

**Lemma 7**

$G(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in $O(\sum_{(e,s) \in G(R)} \deg_{G(R)}(e))$ time

1. Edges of $V(R \cup \{S\})$ which were alreay edges of $V(R)$ don’t changes

2. Edges of $V(R \cup \{S\})$ which are parts of edges in $L$
   - consider each edge $e \in L$
   - If $e \subseteq Q$, $e$ has to be deleted from conflict graph.
   - If $e \not\subseteq Q$, $e - Q$ consists at most two subsegment.
   - let $e'$ be one of the subsegments and let $t$ be a site in $S \setminus R \cup \{s\}$.
   - $e' \cap \text{VR}(t, R \cup \{s, t\}) = e' \cap D(t, r) \cap D(t, s) = e' \cap \text{VR}(t, R \cup \{t\}) \cap D(t, s) \subseteq e \cap D(t, s)$
   - Any site $t$ in conflict with $e'$ must be in conflict with $e$
   - Takes time $O(\sum_{e \in L} \deg_{G(R)}(e)) = O(\sum_{e,s} \deg_{G(R)}(e))$

3. Edges of $\text{VR}(s, R \cup \{s\})$ which are complete new
   - Let $e_{12}$ connect $x_1$ and $x_2$ in $B$
   - Let $e_{12}$ belong to $\text{VR}(p, R)$ such that $e_{12}$ belongs to $J(p, s)$
   - Let $x_1 \in e_1$ of $\text{VR}(p, R)$ and $x_2 \in e_2$ of $\text{VR}(p, R)$
   - Let $P$ be the part of $\text{bd } \text{VR}(p, R)$ which connects $x_1$ and $x_2$ and is contained in $\text{cl } Q$.
   - Lemma 8 will prove that If $t \in S \setminus R \cup \{s\}$ is in conflict with $e_{12}$, $t$ must be in conflict with either $e_1, e_2$ or one of the edges of $P$
   - Each edge in $L$ is involved at most twice, takes time $O(\sum_{e,s} \deg_{G(R)}(e))$
Lemma 7
Let $t \in S \setminus (R \cup \{s\})$ and let $t$ conflict with $e_{12}$ in $V(R \cup \{s\})$ (as defined in Lemma 7). $t$ conflicts with $e_1$, $e_2$, or one of the edges of $P$.

Proof:
• By the definition of conflict, a point $x \in e_{12}$ exists such that $x \in VR(t, R \cup \{s, t\}) \subseteq VR(t, R \cup \{t\})$
• Assume the contrary that $t$ does not conflict with $e_1$, $e_2$, or one edge of $P$.
• For any sufficiently small neighborhood of $U(x_1)$ of $x_1$, $VR(t, R \cup \{s, t\}) \cap U(x_1) \subseteq VR(t, R \cup \{t\}) \cap U(x_1) = \emptyset$, and it is also true for $x_2$.
• Let $p$ be a site in $R$ such that $e_{12} \subseteq cl\ VR(p, R \cup \{s\})$, implying that $x_1, x_2 \in cl\ VR(p, R \cup \{s\})$
• There is a path $P'$ from $x_1$ to $x_2$ completely inside $VR(p, R \{s, t\}) \subseteq VR(p, R \cup \{t\})$.
• The cycle $x_1 \circ P \circ x_2 \circ P'$ contains $VR(t, R \cup \{t\})$ and is contained in $VR(p, R \cup \{t\})$.
• Contradict $VR(p, R \cup \{t\})$ is simply connected.

\[\text{Theorem 1}\]
Let $s \in S \setminus R$. $G(R \cup \{s\})$ and $V(R \cup \{s\})$ can be constructed from $G(R)$ and $V(R)$ in time $O(\Sigma_{(e,s) \in G(R)} deg_{G(R)}(e))$.
Theorem 2
\( V(S) \) can be computed in \( O(n\log n) \) expected time

- \( \sum_{3\leq i\leq n} O(\sum_{(e,s)\in G(R_{i-1})} \deg_{G(R_{i-1})}(e)) \)
- Let \( e \) be a Voronoi edge of \( V(R_i) \) and let \( s \) be a site in \( S \setminus R_i \) which conflicts \( e \).
- The conflict relation \( (e, s) \) will be counted only once since the counting only occurred when \( e \) is removed.
  - Let \( s_j \) be the earliest site in the sequence which conflicts with \( e \). Then \( (e, s) \) will be counted in \( \deg_{G(R_{j-1})}(e) \)
- Time proportional to the number of conflict relations between Voronoi edges in \( \bigcup_{2\leq i\leq n} V(R_i) \) and sites in \( S \)
- The expected size of conflict history is \( -C_n + \sum_{2\leq i\leq n}(n - j + 1)p_j \)
  - \( C_n \) is the expected size of \( \bigcup_{2\leq i\leq n} V(R_i) \)
  - \( p_j \) is the expected number of Voronoi edges defined by the same two sites in \( V(R_j) \)
- Since \( C_n = O(n) \) and \( p_j = O(1/j) \), the expected run time is \( O(n \log n) \)