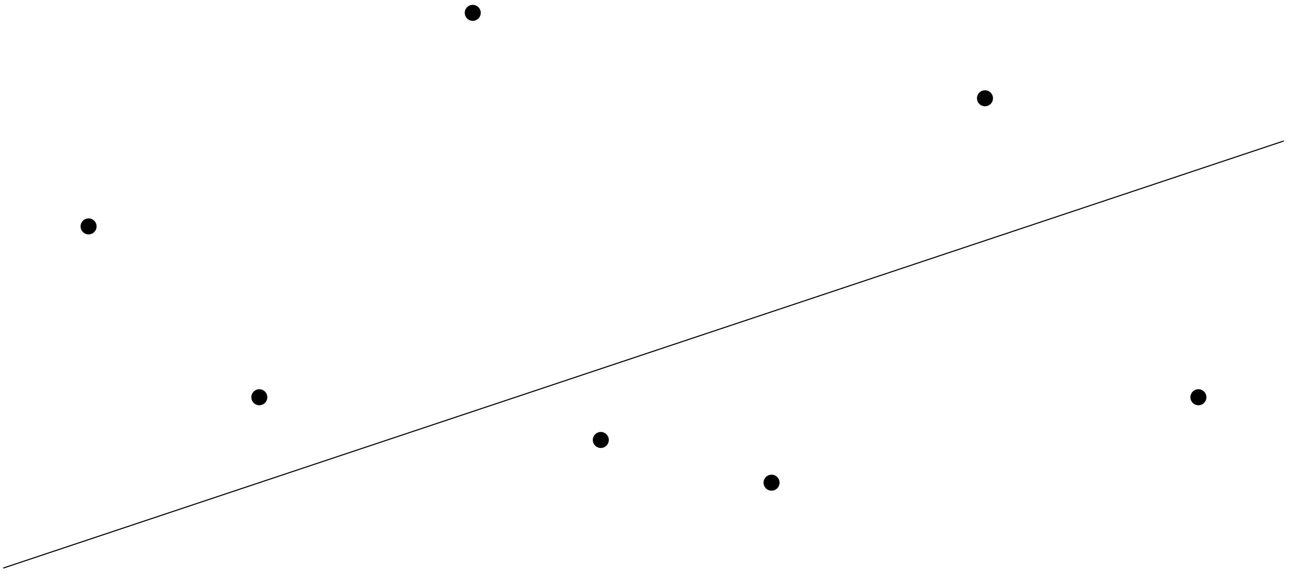
Geometry Duality and k-sets

2-partition

For two subsets A, B of S, A and B form a 2-partition of S if $A \cap B \neq \emptyset$ and $A \cup B = S$.

Given a set S of n points in the plane, how many 2-partitions of S can be separated by a straight line?

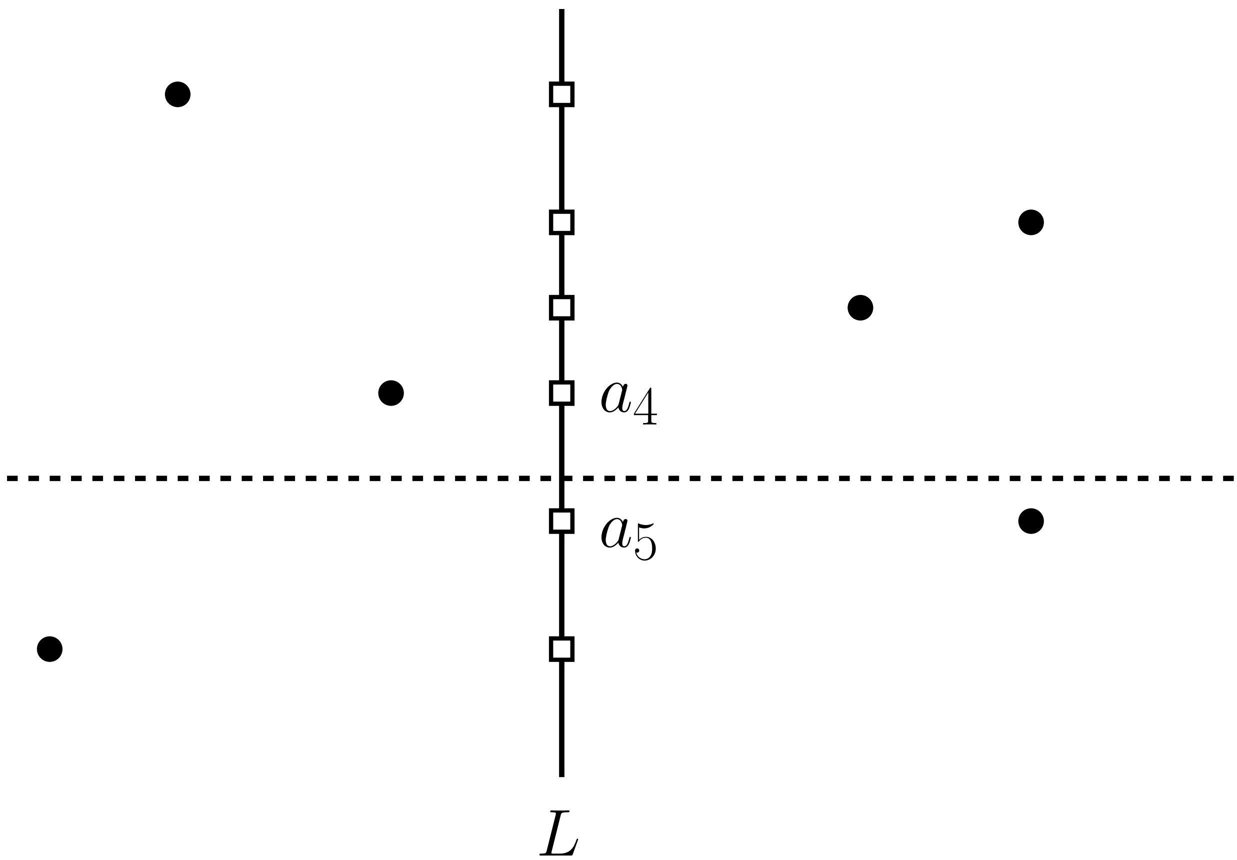


General Position Assumption:

No three points of S are on the same line.

How to count such 2-partitions?

1. Consider a straight L not orthogonal to any line $\overleftrightarrow{pq}$ for any two points $p, q \in S$.
2. Project each point $p \in S$ to L and let p' be the projection point



How to count such 2-partitions?(Continues.)

3. Let $(a_1, a_2, \ldots, a_n)$ be the sequence of projection points on L (in one direction).
4. A straight line orthogonal to L and passing between a_i and a_{i+1} separates S into i-element and $(n-i)$-element subsets.
5. Consider a point c on L and whose y-coordinate smaller than that of all points of S.
6. Rotate L at c coutnerclockwise.
7. When L will be orthogonal to $\overline{pq}$ for two points, $p, q \in S$:
   - Their projection points are adjacent in the sequence of projection points of S on L, i.e., if the projection point of p is a_i, the projection point of q is a_{i+1} or a_{i-1}.
   - When L is orthogonal to $\overleftrightarrow{pq}$, the two projections are coincident, and after that, their positions in the seqeuence are swapped.

How to count such 2-partitions?(Continues.)

8. For $1 \leq i \leq n$, let p_i be a point of S whose projection point on L is a_i
9. Before the positions of a_i and a_{i+1} are swapped, $\{p_1, \ldots, p_{i-1}, p_i\}$ and $\{p_{i+1}, p_{i+2}, \ldots, p_n\}$ is separated by a straight line orthogonal to L and passing between a_i and a_{i+1}.
10. After the positions of a_i and a_{i+1} are swapped, $\{p_1, \ldots, p_{i-1}, p_{i+1}\}$ and $\{p_i, p_{i+2}, \ldots, p_n\}$ is separated by a straight line orthogonal to L and passing between a_i and a_{i+1}.

# of swaps during the rotation is # of 2-partitions of S which can be separated by a straight line.
$\rightarrow \boldsymbol{n(n-1)}$.

How to enumerate the $n(n-1)$ 2-partitions?
An intuitive method

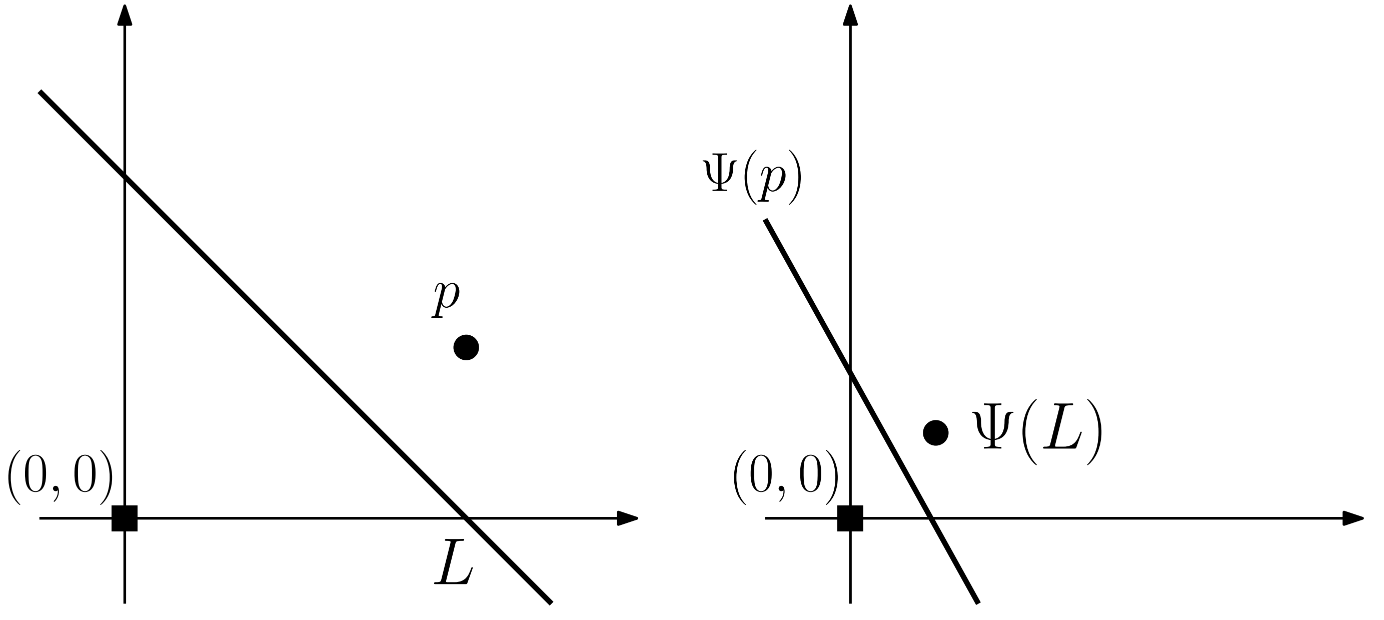
- sort $n(n-1)/2$ slopes of striaght lines passing through two points of S
- Following the order of sorted slopes, compute all the $n(n-1)$ swaps and thus the 2-partitions.
- $O(n^2 \log n)$ time

Can we do better?

- the optimal time is $O(n^2)$
- Using Geometry Duality.

Central Duality Ψ

- For a point $p = (a, b) \in \mathbb{R}^2 \setminus \{0\}$, $\Psi(p)$ is a line: $ax + by = 1$.
- For a line $L : ax + by = 1$, $\Psi(L)$ is a point (a, b).



Fact For a point $p \in \mathbb{R}^2 \setminus \{0\}$ and a line L not passing through the origin, p and the origin are in the same size of L if and only if $\Psi(L)$ and the origin are in the same side of $\Psi(p)$.

Lemma
For a line L not passing through the origin, and a set S of points no of which is the origin, let S_L be the set of points in S which are in the same side of L with the origin, and S_R be the set of points in S which are in the different side of L from the origin.
Then, $\Psi(L)$ and the origin are in the same side of each of $\Psi(S_L)$, but $\Psi(L)$ and the origin are in different sides of each of $\Psi(S_R)$.

Corollary
For a point $p \in \mathbb{R}^2 \setminus \{0\}$, and a set $\mathcal{L}$ of lines no of which passes through the origin, let $\mathcal{L}_L$ be the set of lines in $\mathcal{L}$ each of which includes the origin and p in the same side, and $\mathcal{L}_R$ be the set of lines in $\mathcal{L}$ each of which includes the origin and p in the different sides.
Then, $\Psi(p)$ partitions $\Psi(\mathcal{L})$ into $\Psi(\mathcal{L}_L)$ and $\Psi(\mathcal{L}_R)$.

Theorem
Given a set S of n points, it takes $O(n^2)$ time to generate all the $O(n^2)$ 2-partitions of S which can be separated by a straight line.
Sketch of proof

- Assume no of S is the origin; otherwise translate S.
- Consider the arrangement $A(\Psi(S))$ formed by the n lines in $\Psi(S)$.
- Due to the central duality, for all points p in a cell of $A(\Psi(S))$, $\Psi(p)$ partition S into the same 2-partition.
- For each two adjacent cells in $A(\Psi(S))$, the corresponding two partitons just differ by one point.
- A depth-first-search can visit all $O(n^2)$ cells of $A(\Psi(S))$ in $O(n^2)$ time.

Definition
Given a set S of n points, a subset Q of S is called a k-set if $|Q| = k$ and Q and $S \setminus Q$ can be separated by a straight line.
A $\leq k$-set of S is an i-set of S, $i \leq k$.

Fact
The number of $\leq k$-sets of S is equivalent to the number of switches that occur in the first k positions of the sequence of projection points during the rotation, i.e., the number of switches between a_i and a_{i+1} for $1 \leq i \leq k$.

Theorem
Consider a cyclic sequence of permutations, $P_0,\ P_1, \ldots,\ P_{2N} = P_0$, where $N = \binom{n}{2}$, satisfying

1. P_i and P_{i+N} are in reverse order,
2. and P_{i+1} differs from P_i by an adjacent swtich.

Then the number of swtiches in the first k positions for $2N$ consecutive permutations ist at most kn.
In other words, the number of $\leq k$-sets of n points is at most kn.

Sketch of Proof

- The total number of switches involving an element b is exactly 2n-2 (twice with any other element).
- If b occurs in a switch in position $i \in (1, 2, \dots, k)$, it also occurs in a switch in position $n - i$.
- If $i < j < n-i$, by continuity, b occurs in at least two switches in position j (because b will come back to position i)
- Any element occurs in at most 2n-2-2(n-2k-1)=4k switches in positions $\{1, \ , 2, \dots, \ k\} \cup \{n - k, \ \dots, \ n - 1\}$
- The total number of switches in these positiosn is half of the sum of occurences of elements in such switchesm, i.e., $\leq \frac{1}{2} n 4k = 2nk$.
- The total number of switches for the first k positions is precisely half of this quantity, i.e., $\leq nk$.