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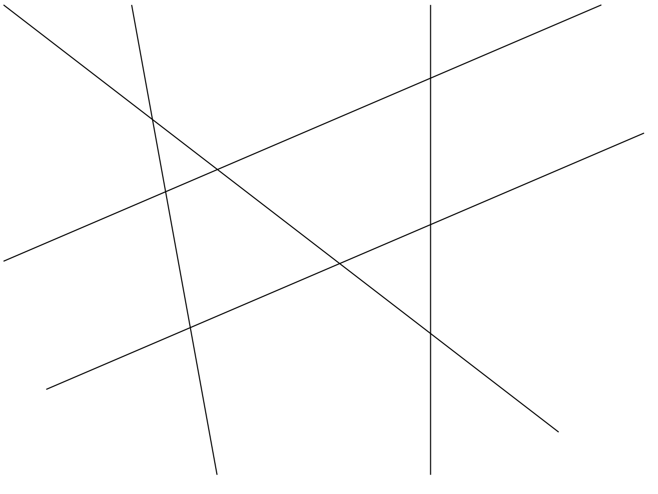
Discrete and **Computational** Geometry

What is discrete geometry?

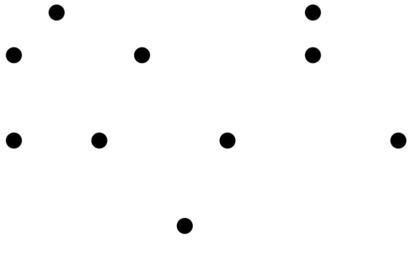
- Discrete sets: points, lines, circles in R^d
- Structural Properties

I. n lines in the plane



Q: How many regions?

II. n points in the plane



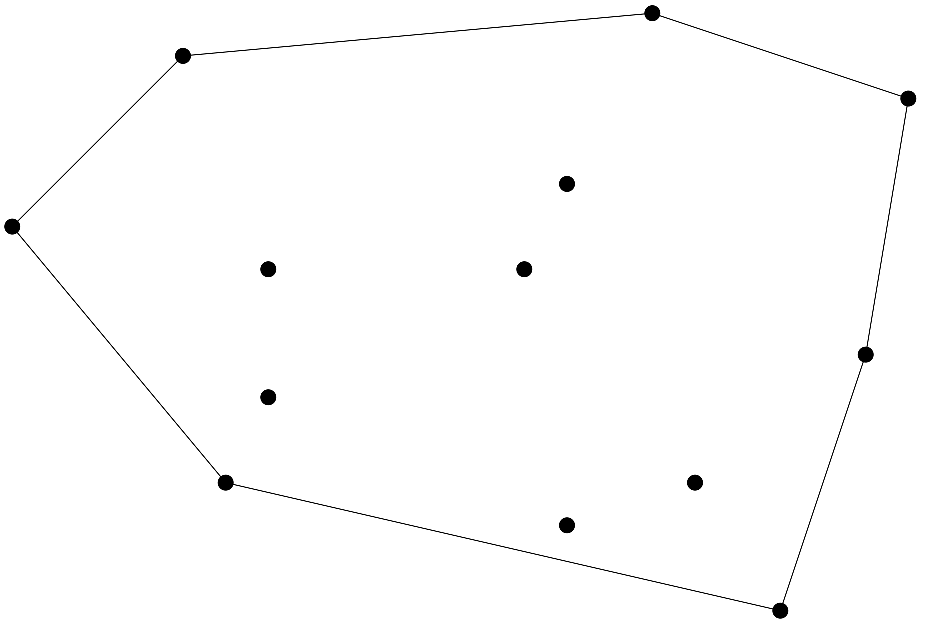
Q: How many of them have the same distance?

Jiři Matoušek, Lecutes on Discrete Geometry

What is computational geometry?

Algorithms for solving geometry problems

Example Convex hulls



Time: $O(n \log n)$

Dynamic Convex hull, 3D convex hull, and convex polytope.

Ketan Mulmuley, Computational Geometry: An Introduction Through Randomized Algorithms

Randomized Incremental Algorithms for Geometry Structure

- Quick Sort and Search
- Vertical Trapezoidal Decomposition
- General Theoretical Foundations
- Dynamic Setting (optionl)

A *probability space* has three components:

1. a sampe space Ω, which is the set of all possible outcomes of the random process modeled by the probability space;
2. a family $\mathcal{F}$ representing the allowable events, where each set in $\mathcal{F}$ is a subset of the sample space; and
3. a probability function $\Pr : \mathcal{F} \to R$ satisfying the following:
   - for any event E, $0 \leq \Pr(E) \leq 1$
   - $\Pr(\Omega) = 1$; and
   - for any finite or countably infinite sequence of pairwise mutually disjoint events E_1, E_2, E_3, $\ldots$,

$$\Pr(\bigcup_{i \geq 1}) = \sum_{i \leq q} \Pr(E_i)$$

Example 1:
One dice

- $\Omega\{1, 2, 3, 4, 5, 6, \}$
- $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{4,5,6\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,3,6\}, \{1,2,4,5\}, \{1,2,4,6\}, \{1,2,5,6\}, \{1,3,4,5\}, \{1,3,4,6\}, \{1,3,5,6\}, \{1,4,5,6\}, \{2,3,4,5\}, \{2,3,4,6\}, \{2,3,5,6\}, \{2,4,5,6\}, \{3,4,5,6\}, \{1,2,3,4,5\}, \{1,2,3,4,6\}, \{1,2,3,5,6\}, \{1,2,4,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}\}$.
  $|\mathcal{F}| = 36$
- $\Pr(\{1\}) = \frac{1}{6}$, $\Pr(\{1,4,5\}) = \frac{1}{2}$, $\ldots$

Example 2:
Two identical dices

- $\Omega = \{\{1,1\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,2\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{3,3\},\{3,4\},\{3,5\},\{3,6\},\{4,4\},\{4,5\},\{4,6\},\{5,5\},\{5,6\},\{6,6\}\}$. ($|\Omega| = 21$)
- $|\mathcal{F}| = 2^{21}$
- $\Pr(\{1,1\}) = \frac{1}{36}$, $\Pr(\{1,4\}) = \frac{1}{18}$, $\Pr(\{\{1,1\},\{1,4\}\}) = \frac{1}{12}$, $\Pr(\{\{1,3\},\{1,4\}\}) = \frac{1}{9}, \ldots$

A *random variable* X on a sample space Ω is a real-valued function on Ω, i.e., $X : \Omega \to R$. A discrete random variable is a random variable that takes on only a finite or countably infinite number of values.

Example 3
Sum of two different dices.

- Let X be the randm variable representing the sum of the two dices.
- $\Pr(X = 4) = \Pr(\{(1,3),(2,2),(3,1)\}) = \frac{1}{12}$

The *expectation* of a discrete random variable X, denoted by $E[X]$, is given by

$$E[X] = \sum_i i\Pr(X = i),$$

where the summation is over all values in the range of X.

Example 3
X is the randm variable representing the sume of the two dices.

$$E[X] = \sum_{2 \le i \le 12} i\Pr(X = i)$$

$$= 2 \times \frac{1}{36} + 3 \times \frac{1}{18} + 4 \times \frac{1}{12} + 5 \times \frac{1}{9} + 6 \times \frac{5}{36} + 7 \times \frac{1}{6} + 8 \times \frac{5}{36} + 9 \times \frac{1}{9}$$
$$+ 10 \times \frac{1}{12} + 11 \times \frac{1}{18} + 12 \times \frac{1}{36} = 7$$

[Linearity of Expectations]:
For n any finite collection of discrete random variable X_1, X_2,, ..., X_n with finite expectations,

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i].$$

Example 4
X is the randm variable representing the sum of the two dices.

- Let X_i be the value of the i^{th} dice. Then $X = X_1 + X_2$.
- $E[X_i] = \sum_{1 \leq i \leq 6} i \times \frac{1}{6} = 3.5$
- $E[X] = E[X_1 + X_2] = E[X_1] + E[X_2] = 7$

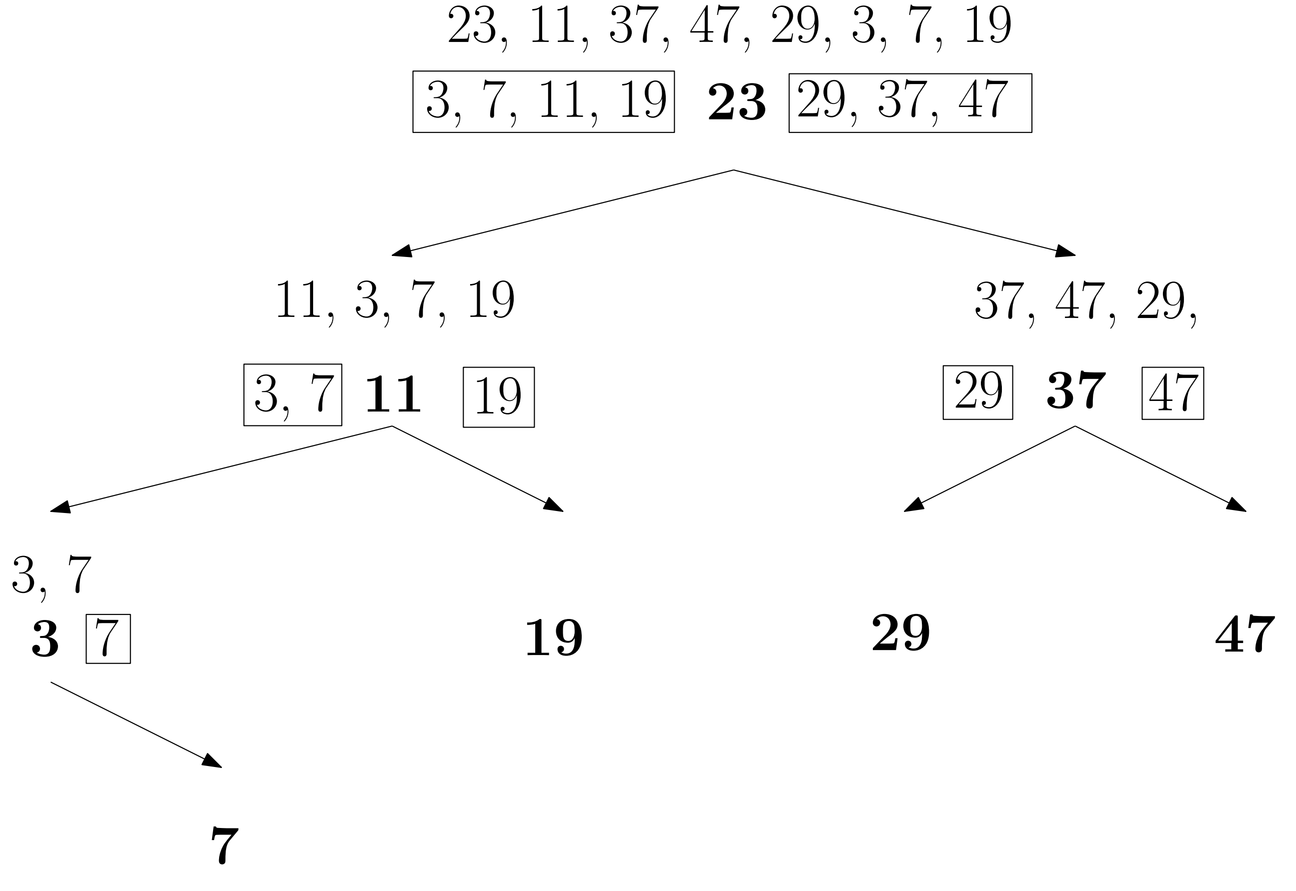
1. **Quick Sort And Search**

Input: a set N of n real numbers (distinct)

Output: an ordered sequence of N

Qucik-Sort(N)

1. If $|N| = 1$, return N.
2. Select a number p from N
3. Let N_L be $\{l \mid l \in N \text{ and } l < p\}$
   Let N_R be $\{r \mid r \in N \text{ and } r > p\}$
4. If $|N_L| > 0$, $L =$ Quick-Sort(N_L); else $L = \emptyset$
5. If $|N_R| > 0$, $L =$ Quick-Sort(N_R); else $R = \emptyset$
6. return a seqeuence L, p, R

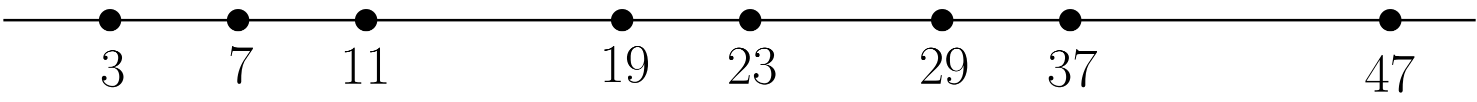


Expected Time Complexity

- If a subset has k elements, it takes $O(k)$ comparisons.
- If a level has m subsets, N_1, N_2, ..., N_m, since they are distinct, a level needs $\sum_{i=1}^{m} O(|N_i|) = O(n)$.
- Expected size of N_L (or N_R)$= \frac{n}{2}$, expected depth of recursion $= O(logn)$
- $O(n \log n)$ expected time

Sorting ⟷ Geometric Structure

An Ordered Seqeuence = A Partition of Real Line R



- **Sorting Problem:**

  Find the partition $\boldsymbol{H(N)}$ of R formed by the given set N of n points.

- **Search Problem:**

Associate a search structure $\widetilde{H}(N)$ with $H(N)$ so that, given any point $q \in R$, one can locate the interval in $H(N)$ containing q quickly, e.g., in logarithmic time.

1.1 Randomized Incremental Version of Quick Sort

$S_1, S_2, \ldots, S_n$: a **random seqeuence** of N

$N^0 = \emptyset$ $N^i = \{S_1, S_2, \ldots, S_i\}$

$H(N^0)$ is R

$H(N^i)$ is the partition of R by N^i

Randomized Incremental Construction:
$H(N^0)$, $H(N^1)$, $H(N^2)$, ……, $H(N^n) = H(N)$.

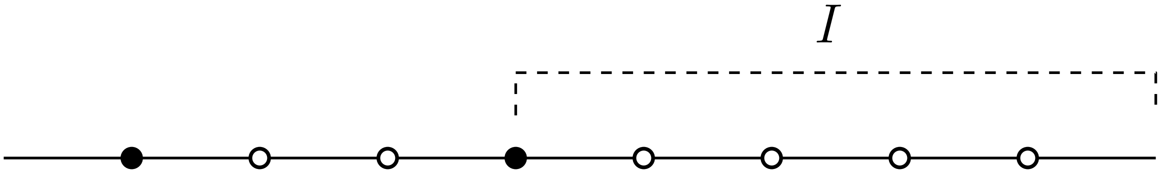


Fig 2. $H(N^2)$ • points in N^2 ∘ points in $N \setminus N^2$

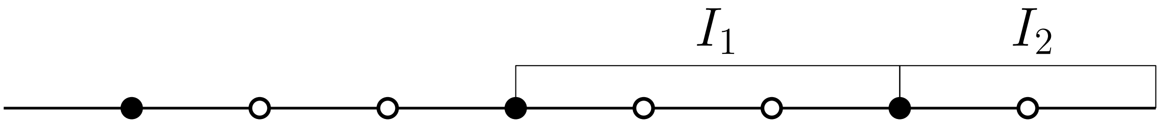


Fig 3. Addition of the third point S^3

Conflict List:
For each interval I in $H(N^i)$, conflict list $\boldsymbol{L(I)}$ is an unsorted list of points in $N \setminus N^i$ contained by I, and $l(I)$ is the size of $L(I)$

E.g., in Fig. 2, $L(I)$ has four points.

Fact
Each point in $N \setminus N^i$ is related to a unique interval in $H(N^i)$.

There is a unique edge between a point in $N \setminus N^i$ and its conflicted interval in $H(N^i)$.

Adding a point $S = S^{i+1}$ into N^i

1. Find a interval I in $H(N^i)$ which contains S.
2. Separate I by S into I_L and I_R.
3. Compute $L(I_L)$ and $L(I_R)$ by $L(I)$

Adding S takes $\boldsymbol{O(l(I_L) + l(I_R) + 1)}$

1. Finding I takes $O(1)$ due to the unique edge between S and I in the conflict list.
2. Separtating I takes $O(1)$ time
3. Computing $L(I_L)$ and $L(I_R)$ takes $O(l(L)) = O(l(I_L)+l(I_R)+1)$ time.

Backward Time Analysis

Inserting S^{i+1} into $H(N^i)$ $=$ Deleting S^{i+1} from $H(N^{i+1})$

Each point S in N^{i+1} is equally likely to be S^{i+1}.
$I_L(S)$: Interval left to S
$I_R(S)$: Interval right to S

Expected Time of Adding S:

$$\frac{1}{i+1} \sum_{S \in N^{i+1}} O(l(I_L(S)) + l(I_R(S)) + 1)$$

$$\leq \quad \frac{2}{i+1} \sum_{J \in H(N^{i+1})} O(I(J) + 1)$$

Each interval are adjacent to at most two points

$$= \quad O(\frac{n}{i+1})$$

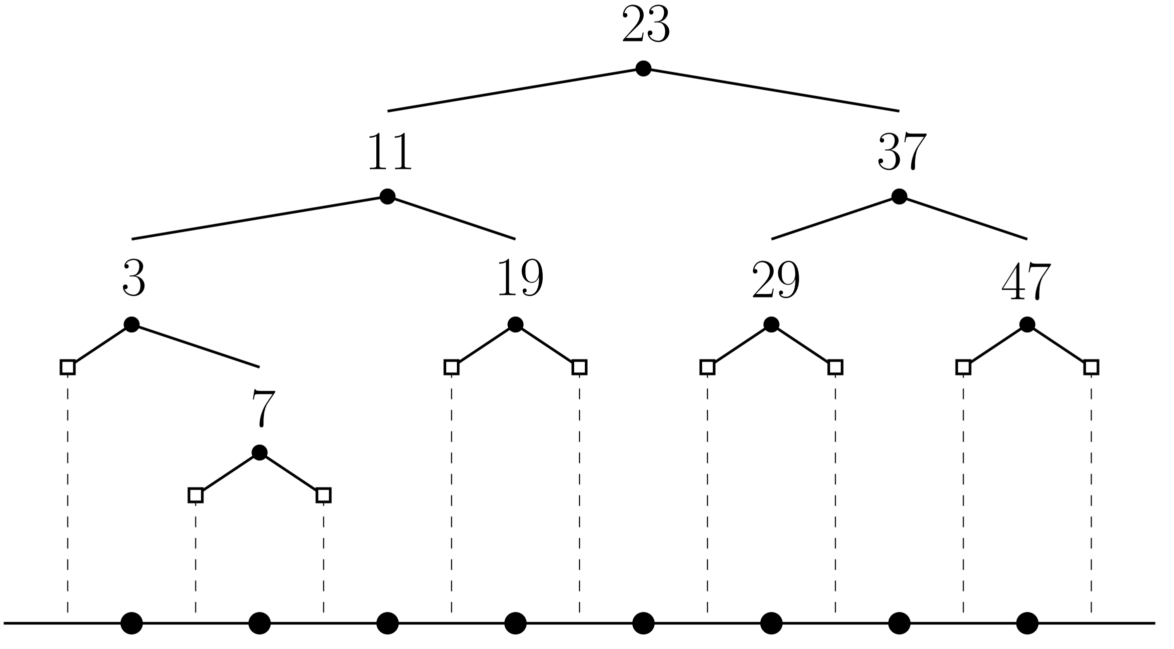
Expected Time Complexity of Randomized Incremental Version:

$$\sum_{i=1}^{n} O(\frac{n}{i+1}) = O(n \log n)$$

1.2 Randomized Binary Tree

$$N = \{\ 23,\ 11,\ 37,\ 47,\ 29,\ 3,\ 7,\ 19\ \}$$
$$S_1\ S_2\ S_3\ S_4\ S_5\ S_6\ S_7\ S_8$$

Divide-and-Conquer Quick-Sort



Random Binary Tree $\widetilde{H}(N)$ is defined as follows:

- If $N = \emptyset$, $\widetilde{H}(N)$ is a node corresponding to the whole real line R
- otherwise,
  - the root of $\widetilde{H}(N)$ is a randomly chosen point $S \in N$
  - $\widetilde{H}(N_L)$ and $\widetilde{H}(N_R)$ are defined recursively for the havles of R on the two sides of S, where N_L and N_R are the sets of points in $N \setminus S$ left to and right to S, respectively.

Search Problem:
Given a point $q \in R$, we locate the invertval in $H(N)$ containing q by applying a binary search on $\widetilde{H}(N)$.

Expected search time = expected depth of $\widetilde{H}(N) = O(\log n)$

1.3 History (On-Line)

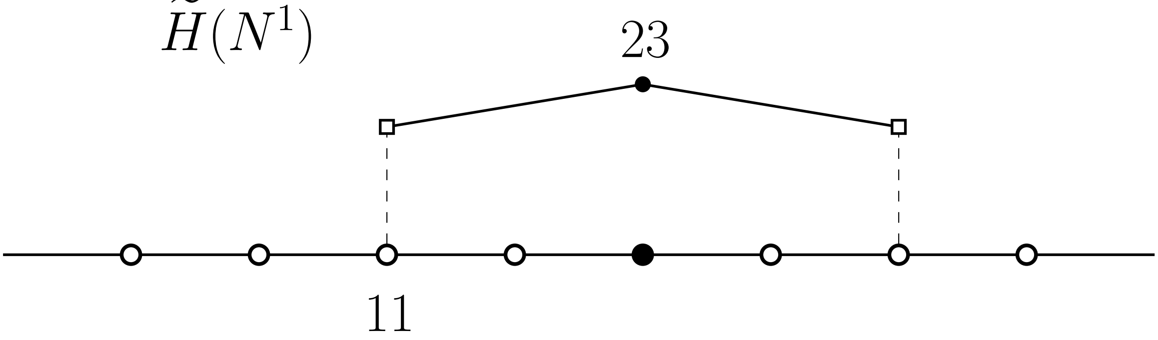
Randomized Incremental Version of Quick-Sort
through the Random Binary Tree

- Locating the interval using the binary tree

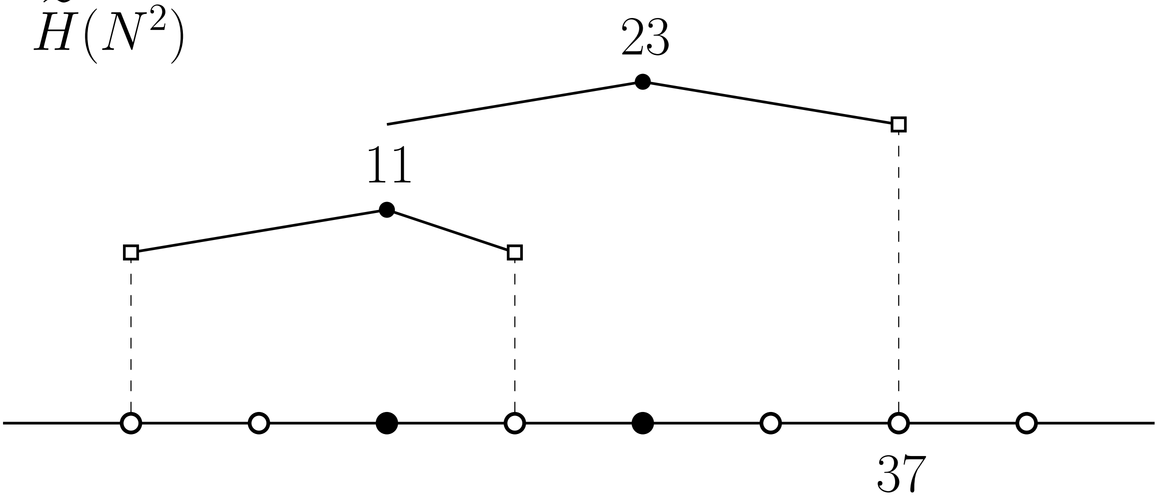
$S_1, S_2, \ldots, S_n$ is a random seqeuence of N

(23, 11, 37, 47, 29, 3, 7, 19)

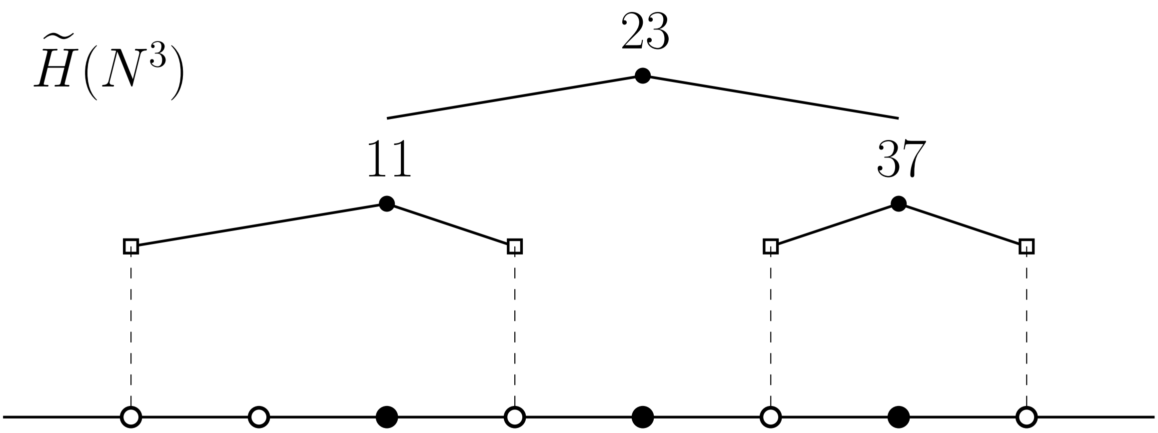
$\widetilde{H}(N^1)$

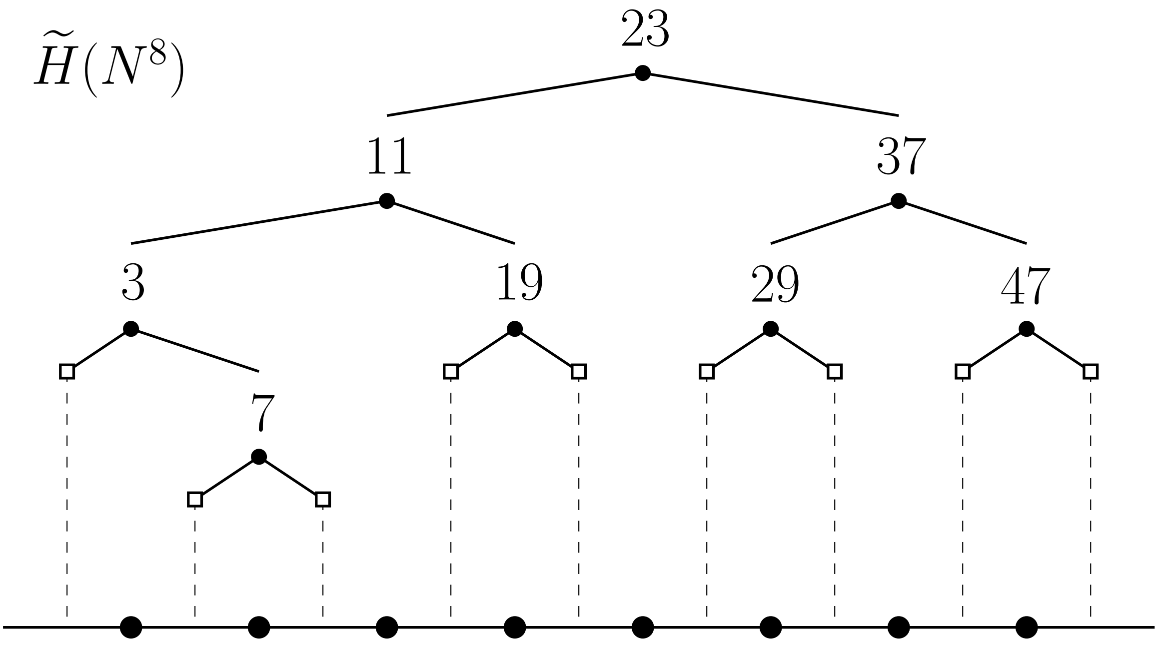


$\widetilde{H}(N^2)$



$\widetilde{H}(N^3)$





Property: If S_j is the left child of S_i, S_j must belong to the left Interval of S_i in $H(N^i)$.

Cost of Inserting S_j = Searching which interval S_j is located in

= Length of Search Path

Backward Analysis

For a query pint q, the search cost is analyzed as follows:

- If the search tests S_i,
  q must belong to the left or right interval of S_i in $H(N^i)$
  $\rightarrow$ probability of testing S_i is $2/i$
- Expected length of search path is $\sum_{i=1}^{n} 2/i = O(\log n)$
- Similarly, inserting S_i takes $O(\log i)$ time

Total Time of Constructing $\widetilde{H}(N)$:

$$\sum_{i=1}^{n} O(\log i) = O(n \log n)$$

This randomized incremental construction through a random binary tree does not require conflict lists:

An on-line algorithm

history(i)

- $\widetilde{H}(N^i)$
- Auxiliary Information
  - Each internal node of $\widetilde{H}(N^i)$ records the left and right intervals when it was created.
  - Each interval records the creation and the deletion time (if it is dead).

history(i)

- Contains the entire history of construction, $\widetilde{H}(N^0), \widetilde{H}(N^1), \ldots, \widetilde{H}(N^n)$.
- Allow searching in $\widetilde{H}(N^i)$ by the auxiliary information.