6. Construction of AVD

Finite Part of AVD

• Let \( \Gamma \) be a simple closed curve such that all intersections between bisectring curve lie inside the inner domain of \( \Gamma \)
• Consider a site \( \infty \), define \( J(p, \infty) = J(\infty, p) \) to be \( \Gamma \) for all sites \( p \in S \), and \( D(\infty, p) \) to be the outer domain of \( \Gamma \) for all sites \( p \in S \).

Incremental Construction

• Let \( s_1, s_2, \ldots, s_n \) be a random sequence of \( S \)
• Let \( R_i \) be \( \{\infty, s_1, s_2, \ldots, s_i\} \)
• Iteratively construct \( V(R_2), V(R_3), \ldots, V(R_n) \)

General Position Assumption

• No \( J(p, q), J(p, r) \) and \( J(p, t) \) intersect the same point for any four distinct sites, \( p, q, r, t \in S \)
  \( \rightarrow \) Degree of a Voronoi vertex is 3

Remark

• For \( 1 \leq i \leq n \) and for all sites \( p \in R_i \), \( VR(p, R_i) \) is simply connected, i.e., path connected and no hole
• If \( J(p, q) \) and \( J(p, r) \) intersect at a point \( x \), \( J(q, r) \) must pass through \( x \)
Basic Operations

- Given \( J(p, q) \) and a point \( v \), determine \( v \in D(p, q) \), \( v \in J(p, q) \), or \( v \in D(q, p) \)
- Given a point \( v \) in common to three bisecting curves, determine the clockwise order of the curves around \( v \)
- Given points \( u \in J(p, q) \) and \( w \in J(p, r) \) and orientation of these curves, determine the first point of \( J(p, r) \) crossed by \( J(p, q) \) \( v \) on \( J(p, q) \) \( x \)
- Given \( J(p, q) \) with an orientation and points \( v, w, x \) on \( J(p, q) \), determine if \( v \) comes before \( w \) on \( J(p, q) \) \( x \)

Notation: Give a connected subset \( A \) of \( R^2 \), int\( A \), bd\( A \), and cl\( A \) mean the interior, the boundary, and the closure of \( A \), respectively.

Conflict Graph \( G(R) \), where \( R \) is \( R_i \) for \( 2 \leq i \leq n \)

- bipartitle graph (U, V, E)
- U: Voronoi edges of \( V(R) \)
- V: Sites in \( S \setminus R \)
- \( E : \{ (e, s) | e \in V(R), s \in S \setminus R, e \cap VR(s, R \cup \{s\}) \neq \emptyset \} \)
  - a conflict relation between \( e \) and \( s \).

Remark:

a Voronoi edge is defined by 4 sites under the general position assumption
Lemma 1
Let $R \subseteq S$ and $t \in S \setminus R$. Let $e$ be the Voronoi edge between $\text{VR}(p, R)$ and $\text{VR}(q, R)$. $e \cap \text{VR}(t, R \cup \{t\}) = e \cap R(t, \{p, q, r\})$. (Local Test is enough)

Proof:

$\subseteq$: Immediately from $\text{VR}(t, R \cup \{t\}) \subseteq \text{VR}(t, \{p, q, t\})$

$\supseteq$: Let $x \in e \cap \text{VR}(t, \{p, q, t\})$

- Since $x \in e, x \in \text{VR}(p, R) \cup \text{VR}(q, R)$ and $x \notin \text{VR}(r, R) \supseteq \text{VR}(r, R \cup \{t\})$ for any $r \in R \setminus \{p, q\}$.
- Since $x \in \text{VR}(t, \{p, q, t\}), x \notin \text{VR}(p, \{p, q, t\}) \cup \text{VR}(q, \{p, q, t\}) \supseteq \text{VR}(p, R \cup \{t\}) \cup \text{VR}(q, R \cup \{t\})$
- $x \notin \text{VR}(r, R \cup \{t\})$ for any site $r \in R \rightarrow x \in \text{VR}(t, R \cup \{t\})$

Lemma 2
$\text{cl} e \cap \text{cl} \text{VR}(s, R \cup \{s\}) \neq \emptyset$ implies $e \cap \text{VR}(s, R \cup \{s\}) = \emptyset$

proof

- Let $x$ belong to $\text{cl} e \cap \text{cl} \text{VR}(s, R \cup \{s\})$
- $x$ is an endpoint of $e$:
  - $x$ is the intersection among three curves in $R$
  - For any $r \in R, J(s, r)$ cannot pass through $x$ due to the general position assumption
  - $x \in D(s, r) \rightarrow$ the neighborhood of $x \in D(s, r)$
  - $\exists y \in e$ belongs to $\text{VR}(s, R \cup \{s\})$
- $x \in e \cap \text{bd} \text{VR}(s, R \cup \{s\})$
  - $x \in J(p, q) \cap J(s, r)$
  - a point $y \in e$ in the neighborhood of $x$ such that $y \in \text{VR}(s, R \cup \{s\})$
Let $Q$ be $\text{VR}(s, R \cup \{s\})$

**Lemma 3**

$Q = \emptyset$ if and only if $\deg_{G(R)}(s) = 0$

**Proof (→)** If $Q = \emptyset$, $\deg_{G(R)}(s) = 0$

(←)

- $\deg_{G(R)}(s) = 0$ implies $\text{cl } Q \subseteq \text{int } \text{VR}(r, R)$ for some $r \in R$
- $\text{VR}(r, R \cup \{s\}) = \text{VR}(r, R) - Q$
- Since $\text{VR}(r, R \cup \{s\})$ must be simply connected, $Q = \emptyset$

**Lemma 4**

Let $I$ be $V(R) \cap \text{bd } Q$.

$I$ is a connected set which intersects $\text{bd } Q$ in at least two points.

**Proof:**

- $\text{bd } Q$ is a closed curve which does not go through any vertex of $V(R)$ due to the general position assumption.
- Let $I_1, I_2, \ldots, I_k$ be connected components of $I$
- Claim: $I_j$, $1 \leq j \leq k$, contains two points of $\text{bd } Q$.
  - If $I_j$ contains no point, $I_j \subseteq \text{int } Q$. In other words, for some $r \in R$, $\text{VR}(r, R)$ contains $I_j$, contradicting that $\text{VR}(r, R)$ must be simply connected
  - If $I_j$ intersects exactly one point $x$ on $\text{bd } Q$, let $e$ be the Voronoi edge of $V(R)$ which contains $x$. Then both sides of $e$ belong to the same Voronoi region. There exists a contradiction.
Assume the contrary that $k \geq 2$

- There is a path $P \subseteq \text{cl } Q - (\bigcup_{1 \leq j \leq k} I_j)$ connects two points on $\text{bd } Q$ such that one component of $Q - P$ contains $I_1$ and the other component contains $I_2$.

- Let $x, y$ be the two endpoints of $P$ and let $r \in R$ such that $P \subseteq \text{VR}(r, R)$.

- Since $x, y \notin V(R)$, $\text{VR}(r, R \cup \{s\}) = \text{VR}(r, R) - Q \neq \emptyset \rightarrow x, y \in \text{cl VR}(r, R \cup \{s\})$

- Since $x, y \in \text{cl VR}(r, R \cup \{s\})$, there is a path $P' \subseteq \text{VR}(r, R \cup \{s\})$ with endpoints $x$ and $y$.

- $P \circ P'$ is contained in $\text{cl VR}(r, R)$ and contains either $I_1$ and $I_2$, contradicting $\text{cl VR}(r, R)$ is simply connected
Lemma 5
Let $e$ be an edge of $V(R)$. If $e \cap Q \neq \emptyset$,

- either $(e \cap Q = V(R) \cap Q$ or $e \cap Q$ is a single component),
- or $e - Q$ is a single component

\[ e \cap Q \]
\[ e - Q \]

Proof

- Assume first $e \cap Q = V(R) \cap Q$
  - Since $V(R) \cap Q$ is connected, $e \cap Q$ is connected
- Assume next $e \cap Q \neq V(R) \cap Q$
  - At least one endpoint of $e$ is contained in $Q$
  - For every point $x \in e \cap Q$, one of the subpaths of $e$ connecting $x$ to
    an endpoint of $e$ must be contained in $Q$
  - $e \cap Q$ or $e - Q$ is a single component

Rough Idea

- Let $L$ be $\{ e \in V(R) \mid (e, s) \in G(R) \}$
- For every edge $e \in L$, let $e'$ be $e - Q = e - \text{VR}(s, R \cup \{s\})$. If $e$ is an
  edge between VR($p, R$) and VR($q, R$), $e' = e - D(s, p) = e - D(s, q)$
- Let $B$ be $\{ x \in x$ is an endpoint of $e'$ but is not an endpoint of $e \} = V(R) \cap \text{bd} Q$
- $\text{bd} Q$ is a cyclic ordering on the points in $B$
**Step 1:** Compute $e'$ for each edge $e \in L$

**Step 2:** Compute $B$ and cyclic ordering on $B$ induced by bd $Q$

**Step 3:** Let $x_1, \ldots, x_k$ be the set $B$ in its cyclic ordering ($x_{k+1} = x_1$), and let $r_i$ such that $(x_i, x_{i+1}) \in VR(r_i, r)$
- For $1 \leq i \leq k$, add the part of $J(r_i, s)$ with endpoints $x_i$ and $x_{i+1}$

**Lemma 6**

$V(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in time $O(\deg_{G(R)}(s) + 1)$

**Lemma 7**

$G(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in $O(\sum_{(e,s) \in G(R)} \deg_{G(R)}(e))$ time

1. Edges of $V(R \cup \{S\})$ which were already edges of $V(R)$ don't change
2. Edges of $V(R \cup \{S\})$ which are parts of edges in $L$
   - consider each edge $e \in L$
   - If $e \subseteq Q$, $e$ has to be deleted from conflict graph.
   - If $e \not\subseteq Q$, $e - Q$ consists at most two subsegment.
   - let $e'$ be one of the subsegments and let $t$ be a site in $S \setminus R \cup \{s\}$.
   - $e' \cap VR(t, R \cup \{s, t\}) = e' \cap D(t, r) \cap D(t, s) = e' \cap VR(t, R \cup \{t\}) \cap D(t, s) \subseteq e \cap VR(t, R \cup \{t\})$
   - Any site $t$ in conflict with $e'$ must be in conflict with $e$
   - Takes time $O(\sum_{e \in L} \deg_{G(R)}(e)) = O(\sum_{(e,s) \in G(R)} \deg_{G(R)}(e))$
3. Edges of $VR(s, R \cup \{s\})$ which are complete new
   - Let $e_{12}$ connect $x_1$ and $x_2$ in $B$
   - Let $e_{12}$ belong to $VR(p, R)$ such that $e_{12}$ belongs to $J(p, s)$
   - Let $x_1 \in e_1$ of $VR(p, R)$ and $x_2 \in e_2$ of $VR(p, R)$
   - Let $P$ be the part of bd $VR(p, R)$ which connects $x_1$ and $x_2$ and is contained in cl $Q$.
   - Lemma 8 will prove that If $t \in S \setminus R \cup \{s\}$ is in conflict with $e_{12}$, $t$ must be in conflict with either $e_1, e_2$ or one of the edges of $P$
   - Each edge in $L$ is involved at most twice, takes time $O(\sum_{(e,s) \in G(R)} \deg_{G(R)}(e))$
Lemma 7
Let \( t \in S \setminus (R \cup \{s\}) \) and let \( t \) conflict with \( e_{12} \) in \( V(R \cup \{s\}) \) (as defined in Lemma 7). \( t \) conflicts with \( e_1, e_2, \) or one of the edges of \( P \).

Proof:

- By the definition of conflict, a point \( x \in e_{12} \) exists such that \( x \in VR(t, R \cup \{s, t\}) \subseteq VR(t, R \cup \{t\}) \)
- Assume the contrary that \( t \) does not conflict with \( e_1, e_2, \) or one edge of \( P \).
- For any sufficiently small neighborhood of \( U(x_1) \) of \( x_1 \), \( VR(t, R \cup \{s, t\}) \cap U(x_1) \subseteq VR(t, R \cup \{t\}) \cap U(x_1) = \emptyset \), and it is also true for \( x_2 \).
- Let \( p \) be a site in \( R \) such that \( e_{12} \subseteq cl \ VR(p, R \cup \{s\}) \), implying that \( x_1, x_2 \in cl \ VR(p, R \cup \{s\}) \)
- There is a path \( P' \) from \( x_1 \) to \( x_2 \) completely inside \( VR(p, R \{s, t\}) \subseteq VR(p, R \cup \{t\}) \).
- The cycle \( x_1 \circ P \circ x_2 \circ P' \) contains \( VR(t, R \cup \{t\}) \) and is contained in \( VR(p, R \cup \{t\}) \).
- contradict \( VR(p, R \cup \{t\}) \) is simply connected

![Diagram](image-url)

Theorem 1
Let \( s \in S \setminus R \). \( G(R \cup \{s\}) \) and \( V(R \cup \{s\}) \) can be constructed from \( G(R) \) and \( V(R) \) in time \( O(\sum_{(e,s) \in G(R)} deg_{G(R)}(e)) \)
Theorem 2

$V(S)$ can be computed in $O(n \log n)$ expected time

- $\sum_{3 \leq i \leq n} O(\sum_{(e, s) \in G(R_{i-1})} \deg_{G(R_{i-1})}(e))$

- Let $e$ be a Voronoi edge of $V(R_i)$ and let $s$ be a site in $S \setminus R_i$ which conflicts $e$.

- The conflict relation $(e, s)$ will be counted only once since the counting only occurred when $e$ is removed
  - Let $s_j$ be the earliest site in the sequence which conflicts with $e$. Then $(e, s)$ will be counted in $\deg_{G(R_{j-1})}(e)$

- Time proportional to the number of conflict relations between Voronoi edges in $\bigcup_{2 \leq i \leq n} V(R_i)$ and sites in $S$

- The expected size of conflict history is $-C_n + \sum_{2 \leq i \leq n} (n - j + 1)p_j$
  - $C_n$ is the expected size of $\bigcup_{2 \leq i \leq n} V(R_i)$
  - $p_j$ is the expected number of Voronoi edges defined by the same two sites in $V(R_j)$

- Since $C_n = O(n)$ and $p_j = O(1/j)$, the expected run time is $O(n \log n)$